ON THE TOPIC OF MOTION INTEGRALS

A Thesis

by

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ABSTRACT

An integral of motion is a function of the states of a dynamical system that is constant along the system's trajectories. Integrals are known for their utility as a means of reducing the dimension of a system, effectively leaving only one differential – or in some cases algebraic – equation to be solved. Invariants of dynamical systems have also proven useful in other contexts, such as in estimation, numerical integration and optimal control. Regardless of the manner in which an integral is employed, finding an analytic form for the integrals of a system generally requires solution of a system of non-linear partial differential equations, with the exception of cases in which certain symmetries of the system are apparent.

The objective of this work is to investigate a generalized method for determining motion integrals for non-linear dynamical systems. This method will not work for all nonlinear systems. Indeed, it is expected that the results will test the limitations of this method. In this we consider a method for determining integrals of motion for a small class of dynamical systems akin to the traditional series expansion method for solving partial differential equations. This method involves posing a candidate integral of motion as a series expansion in terms of some set of polynomials. The coefficients of the candidate polynomial are treated as the unknowns in a system of equations. The system of equations is constructed by sampling simulated trajectories of the dynamical system in question. Then the coefficients are solved for using singular value decomposition.

There are a number of parameters that can potentially affect this method's ability to generate an integral of motion that effectively approximates the phase space of the full nonlinear dynamical system. Part of this thesis proposes what some of these

parameters might be and investigates how they affect the outcome. A couple of well-known systems are used to conduct these tests: the simple pendulum and the rotating rigid body. The simple pendulum is one of the simplest examples of a non-linear system, and examples of the rotating rigid body in aerospace engineering are ubiquitous.

DEDICATION

To my parents.

ACKNOWLEDGEMENTS

These acknowledgements are meant as a statement of appreciation not only for the help received while researching and writing this thesis, but also for being part of my life as a graduate student at Texas A&M in both the Department of Physics and the Department of Aerospace Engineering. I may never again have such a forum in which to express my thanks to all of these people so reader, please excuse my verbosity just this once.

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Even surrounded by people, a person can be lonely without friends. I was lucky enough to have a few in College Station. Jim Ferguson was a colleague in physics, and I have had the pleasure of teaching with him on many occasions. I'll miss our conversations over coffee. Jonathan Asaadi was also a colleague in physics. As our work has taken to different places we talk less these days. Nonetheless, I considered him a good friend when we were both in College Station, and I admire his upbeat attitude toward life. I could not have asked for a better flight instructor than Steven Mapel. Flying with him, there are many things about flying, and some things about life, that I will never forget.

My parents have always supported me in anything I sought to do, and this was no exception. They know I am thankful for everything they do, but for the sake of posterity I'll say it again here: "Mom and Dad. Thank you."

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has scarified many hours and days of our time together, she tolerates my love for the airplane (whom she claims is my second wife), and she has provided the best support one can get. I cannot imagine life without her.

The same can be said for our cat Liam. Although I *can* imagine a life with a few less scratches. I am, however, thankful that he has been there for Sarita in my absence, and I am sure that he is happy to have his play toy back when I return.

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1. INTRODUCTION

The first integrals of a dynamical system play a role arguably as fundamental as the equations of motion. An integral of motion is a function of the states of a dynamical system that is constant along the system's trajectories. Integrals are known for their utility as a means of reducing the dimension of a system, effectively leaving only one differential – or in some cases algebraic – equation to be solved. Invariants of dynamical systems have also proven useful in other contexts, such as in estimation, numerical integration and optimal control. Regardless of the manner in which an integral is employed, with the exception of cases in which certain symmetries of the system are apparent, finding an analytic form for the integrals of a system generally requires solution of a system of non-linear partial differential equations.

This thesis considers a method for determining integrals of motion for a small class of dynamical systems akin to the traditional series expansion method for solving partial differential equations. This method involves making what is, in the end, an approximation to arrive at a candidate integral of motion. This is a compromise that we have come to accept because not all systems are completely integrable, but all systems are at least locally integrable. The efficacy of this method is tested on a couple of well-known systems: the simple pendulum and the rotating rigid body. The simple pendulum is one of the simplest examples of a non-linear system and has served as a model in many real-world applications, and examples of the rotating rigid body in aerospace engineering are ubiquitous.

This method requires that an *ansatz* (or educated guess), $\tilde{\psi}$, be made for the can-

didate integral. Determining the coefficients of the *ansatz* subject to the constraint

$$\frac{d\tilde{\psi}}{dt} = 0, \tag{1.1}$$

is the objective. One method seeks to solve for the coefficients by sampling the state space in order to build an under- or over-determined system of equations. Of course we cannot assume that this method will be applicable throughout all phase space of a given dynamical system for all dynamical systems, and indeed we do not. Part of this investigation entails mapping out the extent to which these methods are valid.

The remainder of this chapter is organized as follows: Section 1.1 briefly surveys the history of the problem of determining integrals of motion[1], and Section 1.2 discusses the utility of motion integrals for control, estimation and numerical integration.

The remainder of this thesis is organized as follows: Chapter 2 will cover some background on the Killing equations, integrability, and chaos, and how they pertain to the methods and dynamical systems discussed herein. In Chapter 3, the proposed methods is explained and applied, and results are discussed.

1.1 Relevance

There is no truly general method for calculating motion integrals. Ultimately one must either solve a set of partial differential equations, or circumvent them entirely. The closest example there is to general method is application of Noether's theorem. It was Emmy Noether [1, 2] who first made the connection between global symmetries and conserved quantities of a system. The theorem that bears her name states that for systems describable by a Lagrangian, there exists a conserved quantity corre-

[1]Throughout this thesis *first integral*, *integral of motion*, *invariant*, and *motion constant* will all be used interchangeably. Although a distinction is often made between a motion constant and an integral of motion, we will not distinguish between the two here.

sponding to each one-parameter group that leaves the action invariant. In some cases these symmetries are known or apparent because they are geometric symmetries of the system, knowledge of which allows one to forego solving a set of partial differential equations. However, if one does not know the group of transformations *a priori*, one must then solve the Killing equations to determine the group of transformations under which the action is invariant.

Many non-linear systems are not completely integrable, which is to say that they do not possess a complete set of isolating integrals. The restricted three-body problem, for example, belongs to the class of non-integrable systems with two degrees of freedom. The simpler circular restricted three-body problem possesses two analytic integrals. Hénon and Heiles [3] presented numerical results indicating that a "third" integral of the circular restricted three-body problem exists for a finite set of initial conditions. Observations of velocity distributions of stars in the galaxy seemed to support the claim. Prior to that Contopoulos [4] had devised a method to construct the third integral for a galaxy-type potential as a series expansion, which was subsequently applied to the restricted three-body problem [5]. This method utilizes the fact that the Poisson bracket of any two independent first integrals vanishes. Since Poincaré had proven that the three-body is non-integrable, this integral is only "local" in the sense that only in certain regions of the phase space does it behave like an isolating integral. The restricted three-body problem is capable of demonstrating chaotic behavior, so only for a subset of the initial conditions and parameters can there exist a third isolating integral.

The utility of Noether's theorem has been the motivation for many attempts at extending the theorem beyond its context. Hojman [6] extends Noether's theorem by considering invariances of the equations of motion rather than of the action integral. The conservation laws that follow from "Noetherian" symmetries are generally only

a subset of all conserved quantities of a system. This is related to the fact that, according to Noether's theorem, each one-parameter group that leaves the action invariant is associated with a conserved quantity. Since invariance of the action is a stronger constraint than invariance of the equations of motion, the number of transformations that satisfy the former is fewer than that of the latter. Indeed, Lutzky [7] finds that, for the harmonic oscillator for example, even though the equations of motion are invariant under the eight-parameter group $SL(3,\mathbb{R})$, only a five-parameter subgroup satisfies the condition of invariance of the action.

Some approaches rely on whether one can find a good set of variable transformations to reduce the burden of solving a set of partial differential equations. Crespo Da Silva [8] considers an approach to finding first integrals of systems that are expressible in Hamiltonian form for the case in which no "classical integrals" exist. In this context, classical integrals refer to the conjugate momenta of cyclic (*i.e.*, ignorable) coordinates. The method consists of finding a suitable canonical transformation of the coordinates such that the transformed (canonical) Hamiltonian is manifestly time-separable. Of course, there is a set of differential equations that dictate what transformations are allowable. In any case, then one integral of the system can be identified as the part of the transformed Hamiltonian that does not depend explicitly on time.

Jones and Ames [9] sought a method for finding first integrals that was applicable with some generality. The method entails formulating the problem of finding a first integral as a single linear partial differential equation, then reducing the number of independent variables via similarity transformations, and finally solving the resulting partial differential equation using standard methods. Both methods considered in this thesis bear some resemblance to that of Jones and Ames.

1.2 Applications

Miller[10, 11] and independently Nacozy [12] have considered the use of integrals in numerical integration of the N-body problem in a predictor-corrector scheme. These applications of the integrals of motion directly probe the primacy of the equations of motion. Due to limitations of numerical precision or round-off, numerical integration techniques such as Runge-Kutta do not implicitly respect conservation equations should they exist for the system in question. Symplectic integrators for conservative systems, for example, enforce the conservation of energy which results in better accuracy for long-time integrations. Symplectic integrators are one of a group of methods to perform numerical integration called geometrical integrators. The utility of geometric integrators is their ability to preserve certain 'geometric' structures of a set of differential equations that may have important physical applications such as first integrals, symmetries, symplectic structure or phase space volume. Of those integrators that preserve integrals of motion, not all utilize the explicit form of the integrals. Projection methods constitute a sub-class of geometric integrators that explicitly enforce a known constant of motion. A thorough survey of geometric integrators is given in [13]. Similar methods have been utilized in problems of estimation.

The first incarnation of the multiplicative Kalman filter was as a sequential estimator for quaternion attitude coordinates. The multiplicative filter implicitly enforces the normalization constraint of the attitude quaternion, whereas a general extended Kalman filter which uses an additive error model and generally does not preserve invariances of the system. Bonnabel *et al.* [14] have extended the concept of the multiplicative filter to a class of invariant filters which inherently preserves some quantity or symmetry of the system.

Integrals of motion have proven to be useful for optimal control as well. Conserved quantities can be used to lower the order of dynamical systems and hence make way for a simpler solution to an optimal control problem. They can also be used to decompose systems in terms of lower-dimensional subsystems in order to analyze stability and controllability [15, 16, 17]. For example, Berbyuk [18, 19] proves that from a first integral of a homogenous nonlinear system, one can construct an energy-optimal admissible control if certain conditions are satisfied.

Torres [20] has considered formulations of Noether's theorem for optimal control. Noether's theorem as it was originally conceived applies to Euler-Lagrange extremals, whereas in the optimal control theory context one is concerned primarily with Pontryagin extremals. Subsequently, Gouveia and Torres [21] devised a way to automatically compute conservation laws of optimal control problems using a computer algebra system.

2. INTEGRABILITY

In this chapter we discuss the notion of integrability, how it is related to chaos, and the Killing equations. Integrability has been shown to be linked with the existence of integrals of motion. The objective of this chapter is not to discuss integrability rigorously, but rather to untangle the many meanings of the term 'integrable' and to discuss how integrability of dynamical systems relates to conservation laws. For a more thorough discussion of integrability see [22] and [23].

Application of Noether's theorem is one of the best known ways of deriving conserved quantities of a given system and is intimately connected to integrability. Since no discussion of integrability would be complete without mention of Noether's theorem, we briefly discuss the topic. It is a topic to which entire volumes have been dedicated, so it will be discussed only insofar as it relates to the topic at hand. For a more complete and thorough discussion of Noether's theorem see, *e.g.*, [2, 24].

2.1 Complete Integrability & Isolating Integrals

The term 'integrability' is widely applied outside of the context in which it first came about, therefore, some discussion will prove helpful to clarify the meaning of the word. In its original incarnation, the term *integrability* referred to the 'solvability' of a system of differential equations. That is, when a system of differential equations was thought to be solvable by the method of reduction to quadratures, then such a system was considered to be *integrable*. Reduction to quadratures of a system of n differential equations is the solution of that system by n independent integrations, which is generally facilitated by $n-1$ conservation laws that eliminate $n-1$ variables, followed by a single final integration. Generally a system solvable by reduction to quadratures is *Lie-Jacobi integrable* or *completely integrable*, although in order to

prove integrability explicit analytic forms of a system's integrals are not necessary. The motion of integrable dynamical systems is characterized as regular, as opposed to chaotic motion characteristic of some non-integrable systems.

In general, a dynamical system with n degrees of freedom can have at most $2n-1$ constants of motion since there are $2n$ initial conditions and initial time cannot be determined by a constant of motion. The explicit solutions when inverted yield the initial conditions as invariant functions of the system variables. Invertibility, however, is a non-trivial condition.

The inverse function theorem is a special case of the implicit function theorem that gives sufficient conditions for a function to be invertible in the neighborhood of some point at which the function's derivative is non-zero. Specifically, the inverse function theorem states that if a multivariate function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on some open set containing $\boldsymbol{a} \in \mathbb{R}^n$, and if that function's Jacobian determinant is non-zero at $\boldsymbol{a}$, then there exists some open set V containing $\boldsymbol{a}$ and some open set W containing $f(\boldsymbol{a})$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ that is differentiable for all points in W [25].

An integral expressed as a conserved quantity

$$G(x_1, \ldots, x_n) = C, \tag{2.1}$$

can be inverted to

$$x_j = f(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, C). \tag{2.2}$$

If Equation (2.1) can be inverted to yield (2.2), then the conservation law is called an *isolating integral*. The conditions for a successful inversion are dictated by the inverse function theorem.

The idea of foliation of the phase space is useful for understanding isolating and non-isolating integrals. The phase space of a completely integrable system is *foliated* by surfaces to which the trajectories of the system are confined. Consider Euler's top as an example of a completely integrable system. The three-dimensional angular momentum phase space is foliated into concentric spheres as a result of angular momentum conservation, so for any initial value of angular momentum motion of the system is confined to a 'leaf' of the foliation. Energy conservation further confines the system trajectories to lines on the surface of each sphere.

So an integrable flow is one whose solution is reducible to n independent integrations by a single coordinate transformation, which is determined by conservation laws *that hold for all time*. Conversely, a non-integrable system is one for which such transformations cannot, in principle, be carried out for all times. If the dynamics of the system dictate that the solution goes to infinity in finite time, then there exists no coordinate transformation that can be carried out over a time interval including the time at which a singularity occurs.

2.2 Hamiltonian Systems

Hamiltonian systems are discussed here because virtually all dynamical systems can be formulated as a Hamiltonian system. Also, the concept of integrability finds its most natural expression in the Hamiltonian formalism.

A *Hamiltonian system* with n degrees of freedom is a system of $2n$ first-order ordinary differential equations which, when expressed in the symplectic form, are written as

$$\dot{z}_i = J\nabla_{z_i}\mathcal{H}(z_i,t), \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad i = 1,\ldots,2n. \tag{2.3}$$

When written in terms of the configuration variables q and their conjugate momenta

p, the differential equations for a Hamiltonian system are given by the familiar Hamilton equations,

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j}, \quad j = 1, \ldots, n. \tag{2.4}$$

An integrable Hamiltonian system must possess a maximal set of Poisson commuting invariants, which are functions on the phase space that are in mutual involution with each other as well as with the Hamiltonian. A Hamiltonian system with n degrees of freedom has at most n independent Poisson invariants. The connection between Poisson commuting invariants and integrability can be seen as follows.

The Poisson bracket is defined as

$$[F, G] \equiv \sum_{\sigma} \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right),$$

for two functions of the generalized coordinates. The Poisson bracket plays a fundamental role in the time evolution of Hamiltonian systems. This can be seen by considering the time derivative of a general function F of the coordinates and their conjugate momenta

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i = \frac{\partial F}{\partial t} + [F, \mathcal{H}].$$

Suppose some function, $f(p, q)$, is a constant of motion in the Liouville sense, which implies that, if $\{q(t), p(t)\}$ is a solution to the Hamilton equations of motion, then $df/dt = 0$ along that trajectory. It follows that

$$0 = \frac{d}{dt} f(p, q) = \frac{\partial f}{\partial q}\frac{dp}{dt} + \frac{\partial f}{\partial q}\frac{dq}{dt} = [f, \mathcal{H}],$$

so all of the constants of motion must be in involution with the Hamiltonian. They

must also all be in mutual involution which follows from the Jacobi identity.

Equivalently, an integrable Hamiltonian system can, by definition, be transformed to a canonical coordinate system in which the Hamiltonian depends solely on the transformed momenta. From the Hamilton equations it follows that, if the Hamiltonian is dependent solely upon the conjugate momenta, then those conjugate momenta are constants. Moreover, since the mapping $(q, p) \mapsto (Q, P)$ is one-to-one, the conjugate momenta are not only constants, but isolating integrals of motion as well.

Action-angle variables $(\theta_i, \mathcal{J}_i)$ are often the coordinates of choice for such Hamiltonian systems. If the Hamiltonian of a system of n degrees of freedom is independent of the angles θ_i, then the Hamilton equations are

$$\dot{\theta}_i = \partial \mathcal{H}(\mathcal{J}_i)/\partial \mathcal{J}_i, \quad \dot{\mathcal{J}}_i = 0, \quad i = 1, \ldots, n.$$

Arnold [**?**] had proven that, provided energy is conserved (*i.e.*, the system is bound) and the new momenta are everywhere independent, it is possible to choose the new momentum variables so that the conjugate configuration variables are periodic. Consequently, the trajectories of a $2n$-dimensional integrable Hamiltonian system are confined to an n-dimensional tori in phase space, called *invariant* or *Liouville tori*.

Trajectories, or orbits, of the system helically wind around the torus with frequencies $\Omega_i = \dot{\theta}_i$. When the ratio of frequencies is rational, the trajectories on the torus are closed and consequently occupy a torus of lower dimension in phase space. In this case the trajectory called *resonant*. Non-resonant trajectories occur when the ratio of frequencies is irrational, resulting in a trajectory on the torus that is not closed and that consequently densely fills the surface of the torus; it is *quasi-periodic*. Motion characterized by non-resonant tori is not necessarily characterized as chaotic, but it can become chaotic. Resonant orbits are not safe from chaos either. If the

orbits are unstable they can degenerate into non-resonant orbits.

A Hamiltonian system may possess functionally-independent (clarify) conservation laws in excess of the maximal number of Poisson invariants, in which case the system is called *superintegrable*. In some cases, the frequencies of two different generalized coordinates are identical. In such cases the motion is called degenerate, which signals that there are additional conserved quantities as in the Kepler two-body problem.

2.3 Chaos & Non-integrability

The term 'chaotic' as it applies to the classification of dynamical system is not precise. There is no universally accepted definition of chaos, but most agree that all chaotic systems appear to have three things in common:

1. Aperiodic long-term behavior: There are no periodic or quasi-periodic, nor trajectories that flow to fixed points as $t \to \infty$.

2. Deterministic: The system has no erratic or noisy inputs. Irregular behavior arises solely from nonlinearities.

3. Sensitive dependence on initial conditions: Nearby trajectories diverge exponentially fast, *i.e.*, the system has at least one positive Lyapunov exponent.

Note that an unstable system is not necessarily chaotic. If a trajectory runs off to infinity as $t \to \infty$, then ∞ is a fixed point.

When a system is not chaotic it is said to be regular, although it can still become chaotic. Whether this occurs depends upon the stability of the system. If, in a system with n degrees of freedom, a particular orbit admits n independent isolating integrals of motion, the orbit is said to be *regular*. The system trajectories are confined to an n-dimensional manifold in phase space.

Non-integrability is a necessary, but not sufficient, condition for chaotic behavior, and the absence of chaos does not prove complete integrability. The are two minimum requirements that must be satisfied by any system capable of demonstrating chaotic behavior: it must possess at least three dynamical degrees of freedom, and it must be a nonlinear system. And to prove complete integrability would in turn prove the impossibility of chaotic behavior, but for $n \geq 3$ it is often more difficult to prove complete integrability than it is to rule out chaos everywhere in phase space. As an example, consider the Lorenz model

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta x + xy, \end{aligned} \tag{2.5}$$

which is known to have enough conservation laws to be completely integrable for the parameter values $(\sigma, \beta, \rho) = (1/2, 1, 0)$, $(1, 2, 1/9)$, and $(1/3, 1, \rho$ arbitrary). The space of parameters for which this system is known to have non chaotic motions is actually much larger: $\rho < \sigma(\sigma + \beta + 3)/(\sigma - \beta - 1)$, where $\sigma - \beta + 1 > 0$, turns out to be much larger than that determined on the basis of complete integrability.

The topic of chaos is broached here because when dealing with nonlinear systems one will inevitably be confronted with chaotic motion. Indeed, we would be remiss were we not to consider the issue since chaotic systems are non-integrable and as such, do not generally possess a full set of motion integrals.

2.4 The Killing Equations

Noether's theorem states that a system described by the action

$$I = \int_{t_1}^{t_2} L[\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t]dt,$$

that is invariant with respect to the infinitesimal transformations

$$\begin{aligned} t &\mapsto t' = t + \varepsilon T \\ \boldsymbol{q}(t) &\mapsto \boldsymbol{q}'(t') = \phi[\boldsymbol{q}(t), \varepsilon], \end{aligned}$$

possesses motion constants C given by

$$\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\frac{\partial \phi}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} - L\right) T - \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\frac{\partial \phi}{\partial \varepsilon} = C.$$

When Noether's theorem is used to determine conserved quantities of a dynamical system, the groups of transformations that leave the system invariant are often known. If one does not know the group of transformations *a priori*, the solutions of the Killing equations give the group of transformations under which the system is invariant. This development of the Killing equations follows closely that given in [24].

Consider the action integral of a dynamical system with no external forces,

$$S(q_1, \ldots, q_n) = \int_{t_0}^{t_1} dt\, L = \int_{t_0}^{t_1} dt\, \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j, \tag{2.6}$$

where the set of coordinates $\{x^1, \ldots, x^n\}$ represent the generalized coordinates, and $g_{ij} = g_{ij}(x^1, \ldots, x^n)$. We seek the group of transformations that leaves the action

invariant. There is a theorem [24] that states the following: a necessary condition for the action S to be invariant up to a divergence under the group of r-parameter transformations

$$\begin{aligned} \bar{t} &= \phi(t, x, \varepsilon) \\ \bar{x}^k &= \psi^k(t, x, \varepsilon), \quad k = 1, \ldots, n, \ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m), \end{aligned} \tag{2.7}$$

is that the Lagrangian and its derivatives satisfy the r identities (hereafter known as invariance identities),

$$\frac{\partial L}{\partial t}\tau_s + \frac{\partial L}{\partial x^k}\xi_s{}^k + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d\xi_s{}^k}{dt} - \dot{x}^k\frac{d\tau_s}{dt}\right) + L\frac{d\tau_s}{dt} = \frac{d\Phi_s}{dt}, \tag{2.8}$$

$(s = 1, \ldots, r)$, where τ_s and $\xi_s{}^k$ are defined by

$$\tau_s(t, x) = \frac{\partial \phi}{\partial \varepsilon^s}(t, x, 0), \quad \xi_s{}^k(t, x) = \frac{\partial \psi^k}{\partial \varepsilon^s}(t, x, 0). \tag{2.9}$$

In this exercise we seek only one-parameter transformations of the form

$$\bar{t} = t, \quad \bar{x}^k = x^k + \xi^k(x)\varepsilon, \tag{2.10}$$

in which case

$$\tau = 0, \quad \xi^k = \xi^k(x), \tag{2.11}$$

and hence, the invariance identities reduce to

$$\frac{1}{2}g_{ij}(\dot{x}^i\delta^j_k + \dot{x}^j\delta^i_k)\left(\frac{\partial \xi^k}{\partial x^n}\dot{x}^n\right) = 0. \tag{2.12}$$

Simplification and renaming of indices gives

$$\frac{1}{2}g_{ik}\frac{\partial\xi^k}{\partial x^n}\dot{x}^i\dot{x}^n + \frac{1}{2}g_{kj}\frac{\partial\xi^k}{\partial x^n}\dot{x}^j\dot{x}^n = \frac{1}{2}\left(g_{ik}\frac{\partial\xi^k}{\partial x^j} + g_{kj}\frac{\partial\xi^k}{\partial x^i}\right)\dot{x}^i\dot{x}^j = 0, \tag{2.13}$$

which implies that

$$g_{ik}\frac{\partial\xi^k}{\partial x^j} + g_{kj}\frac{\partial\xi^k}{\partial x^i} = 0. \tag{2.14}$$

Therefore, in order for the system described by the action Equation (2.6) to be invariant under the set of one-parameter transformations in Equation (2.10), the generators ξ^k must satisfy Equation (2.14). This system of equations is a specific example of the Killing equations. In general, the Killing equations are a set of partial differential equations in the generators τ_s and $\xi_s{}^k$. Provided there exists a solution to this system of partial differential equations, Noether's theorem guarantees the existence of a first integral, which in this case is given by

$$\left(L - \dot{x}^k\frac{\partial L}{\partial\dot{x}^k}\right)\tau_s + \frac{\partial L}{\partial\dot{x}^k}\xi_s{}^k = C. \tag{2.15}$$

2.4.1 The Damped Harmonic Oscillator

The harmonic oscillator is a linear system, but we consider this system here for two reasons. The first is that the damped harmonic oscillator serves as a simple example of a dissipative system for which the integrals of motion are known. That a dissipative system has any integrals at all is sometimes a peculiar notion, perhaps because we are taught that non-conservative systems, such as this one, do not possess any conserved quantities. The second reason is that the lack of any apparent symmetries associated with the geometry of the problem makes this simple system a good example of how solution of the Killing equations and subsequent application

of Noether's theorem produces a first integral.

The simple pendulum system has two equilibrium points; one stable, $\theta = 0$, and the other unstable, $\theta = \pi$. Linearizing around the stable equilibrium, the equation of motion is given by

$$\ddot{\theta} + \frac{g}{l}\theta = 0. \tag{2.16}$$

If one were then to carry out the same process as that in the previous section, one would find the first integral of the system to be

$$\psi = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\omega^2\theta^2 + \beta, \tag{2.17}$$

which is the same result that one would expect to get by linearizing the integral of the nonlinear system up to second-order in θ.

The harmonic oscillator with an added damping force has been a system of considerable interest, particularly as a means to understand integrability and the link between symmetries and conserved quantities. It is an interesting cause perhaps because it is a dissipative system that possesses a known integral of motion. There are a number of a ways to arrive at the result, but tried and true is via Noether's theorem. The following describes the method as given in [24].

The equation of motion for the damped harmonic oscillator is

$$m\ddot{x} + a\dot{x} + kx = 0.$$

We can show that this equation comes from the Lagrangian

$$L(t, x, \dot{x}) = \frac{1}{2}e^{(a/m)t}(m\dot{x}^2 - kx^2). \tag{2.18}$$

Varying the action we have

$$\begin{aligned}\delta S &= \int dt\, e^{(a/m)t}[(m\dot{x}\delta\dot{x} - kx\delta x) + \frac{a}{m}\delta t] \\ &= \int dt\, \frac{d}{dt}\left(-e^{(a/m)t}m\dot{x}\right)\delta x - e^{(a/m)t}kx\delta x \\ &= \int dt\, e^{(a/m)t}(-m\ddot{x} - a\dot{x} - kx)\delta x.\end{aligned}$$

In general, if the action S is invariant under a group of one-parameter transformations

$$\bar{t} = t + \tau(t,x)\varepsilon, \quad \bar{x} = x + \xi(t,x)\varepsilon,$$

then the identities

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x}\xi + \frac{\partial L}{\partial \dot{x}}\left(\frac{d\xi}{dt} - \dot{x}\frac{d\tau}{dt}\right) + L\frac{d\tau}{dt} = 0$$

hold true. For the damped harmonic oscillator we have

$$\frac{\partial L}{\partial t} = \frac{a}{m}L, \quad \frac{\partial L}{\partial x} = -e^{(a/m)t}kx, \quad \frac{\partial L}{\partial \dot{x}} = e^{(a/m)t}m\dot{x}.$$

Substituting, we then have

$$\begin{aligned}\frac{a}{2m}e^{(a/m)t}(m\dot{x}^2 - kx^2)\tau &- e^{(a/m)t}kx\xi + e^{(a/m)t}m\dot{x}\left(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\dot{x} - \dot{x}\frac{\partial\tau}{\partial t} - \dot{x}^2\frac{\partial\tau}{\partial x}\right) \\ &+ \frac{1}{2}e^{(a/m)t}(m\dot{x}^2 - kx^2)\left(\frac{\partial\tau}{\partial t} - \dot{x}\frac{\partial\tau}{\partial x}\right) = 0.\end{aligned}$$

When the coefficients of $\dot{x}^0$, $\dot{x}$, $\dot{x}^2$ and $\dot{x}^3$ are collected and equated to zero, we obtain the generalized Killing equations:

$$\frac{a}{2m}\tau x + \xi + \frac{x}{2}\frac{\partial\tau}{\partial t} = 0$$

$$\begin{aligned} m\frac{\partial \xi}{\partial t} - \frac{k}{2}\frac{\partial \tau}{\partial x}x^2 &= 0 \\ \frac{a}{2m}\tau + \frac{\partial \xi}{\partial x} - \frac{1}{2}\frac{\partial \tau}{\partial t} &= 0 \\ \frac{\partial \tau}{\partial x} &= 0. \end{aligned}$$

From the last equation one can deduce that $\tau = \tau(t)$, and consequently from the second equation that $\xi = \xi(x)$. These four equations then reduce to two ordinary differential equations:

$$\frac{a}{2m}\tau x + \xi + \frac{x}{2}\frac{d\tau}{dt} = 0$$

$$\frac{a}{2m}\tau + \frac{d\xi}{dx} - \frac{1}{2}\frac{d\tau}{dt} = 0.$$

If the second equation is multiplied by x and added to the first, one obtains

$$\frac{a}{m}\tau x + \xi + x\frac{d\xi}{dx} = 0,$$

which implies that $\tau = \text{constant} = c$. In that case,

$$\xi = -\left(\frac{ac}{2m}\right)x.$$

Upon choosing $c = 1$ one obtains the family of translations

$$\bar{t} = t + \varepsilon, \quad \bar{x} = x - \varepsilon\frac{ax}{2m},$$

under which, by construction, the action is invariant. Hence by Noether's theorem there is a first integral of the motion; *i.e.*,

$$(L - \dot{x}L_{\dot{x}})\tau + L_{\dot{x}}\xi = \text{constant},$$

or in this case,

$$(m\dot{x}^2 + kx^2 + ax\dot{x})e^{(a/m)t} = \text{constant}.$$

Crespo Da Silva [8] arrives at the same result in a slightly different manner. The method entails making the Hamiltonian time-separable by a judicious choice of canonical transformation. That is, one must find a transformation $T(t)$ such that

$$\boldsymbol{x}(t) = T(t)\boldsymbol{x}',$$

and such that the transformed Hamiltonian $\tilde{H}(\boldsymbol{x}', t)$ is canonical and time-separable,

$$\tilde{H}(\boldsymbol{x}', t) = g(t)\tilde{H}'(\boldsymbol{x}').$$

It follows that $\tilde{H}'(\boldsymbol{x}')$ is a constant of the motion as it is time-independent. One begins by writing the Hamiltonian for the system, which in this case can be computed from the Lagrangian (2.18),

$$H = p\dot{q} - L = \frac{1}{2m}p^2e^{-ct} + \frac{k}{2}q^2e^{ct}, \tag{2.19}$$

where $c \equiv a/m$, and where the canonical momentum and generalized coordinates are given by

$$p = \frac{\partial L}{\partial \dot{q}} = e^{ct}m\dot{x}, \quad q = x.$$

The transformed Hamiltonian "canonically associated with $\boldsymbol{x}'$" is given by

$$\tilde{H}(\boldsymbol{x}', t) = H(T\boldsymbol{x}', t) - \frac{1}{2}\boldsymbol{x}'^T T^T \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} \dot{T}\boldsymbol{x}', \tag{2.20}$$

where $\boldsymbol{x} = [q \quad p]^T$, for an n-dimensional system. The problem of identifying an integral of motion is then left to determining the transformation $T(t)$ that renders $\tilde{H}$ time-separable. In this case one happens upon the solution by inspection to be

$$T(t) = \begin{pmatrix} e^{-ct/2} & 0 \\ 0 & e^{ct/2} \end{pmatrix},$$

which implies that for the coordinate transformation one has

$$\begin{aligned} q &= e^{-ct/2} q' \\ p &= e^{ct/2} p'. \end{aligned}$$

Then the transformed Hamiltonian is

$$\begin{aligned} \tilde{H}(\boldsymbol{x}', t) &= \frac{1}{2m} p'^2 + \frac{k}{2} q'^2 + T_{11} \dot{T}_{22} q' p' \\ &= \frac{1}{2m} p'^2 + \frac{k}{2} q'^2 + \frac{c}{2} q' p'. \end{aligned}$$

In this case the time separation is trivial: $\tilde{H}$ as a function of the coordinates $\boldsymbol{x}'$ is manifestly independent of time and as such, is a constant of motion. This is in fact the same conclusion reached using Noether's method if one makes the appropriate coordinate transformations:

$$\begin{aligned} \tilde{H} &= e^{-ct} \frac{1}{2m} p^2 + e^{ct} \frac{k}{2} q^2 + \frac{c}{2} qp \\ &= e^{ct} \left(\frac{1}{2} m \dot{x}^2 + \frac{k}{2} x^2 + \frac{a}{2} x \dot{x} \right). \end{aligned}$$

By all appearances this method is easier than solving a set of differential equations, but in reality, the problem was just shuffled from one differential equation to

another. Using Noether's method one has to solve the Killing equations which is a set of partial differential equations. In this case, one must solve a set of ordinary differential equations to determine the matrix T. It just so happens that in both cases, with a bit of aptitude and a good eye, the solution presents itself.

If one works in state space, it turns out, this system can be shown to be separable, allowing one to find an explicitly time-independent form of an integral for this system. Letting $y = \dot{x}$ one has the system of linear ODEs,

$$\dot{x} = y, \quad \dot{y} = -cy - kx, \tag{2.21}$$

from which it follows that

$$(cy + kx)dx + ydy = 0. \tag{2.22}$$

This equation turns out to be separable with the change of variables $u = y/x$,

$$\frac{dx}{x} + \frac{u}{u^2 + cu + k}du = 0. \tag{2.23}$$

Letting $\beta = c^2 - 4k$ one finds three integrals corresponding to whether the system is underdamped, critically damped or overdamped, respectively,

$$\begin{aligned}
\psi &= \log|x| + \frac{1}{2}\log|u^2 + cu + k| - \frac{c}{\sqrt{-\beta}}\tan^{-1}\left(\frac{2u + c}{\sqrt{-\beta}}\right), \quad \beta < 0, \\
\psi &= \log|x| + \log\left|u + \frac{c}{2}\right| + \frac{c}{2u + c}, \quad \beta = 0, \\
\psi &= \log|x| + \frac{1}{2}\log|u^2 + cu + k| - \frac{c}{2\sqrt{\beta}}\log\left|\frac{2u + c - \sqrt{\beta}}{2u + c + \sqrt{\beta}}\right|, \quad \beta > 0.
\end{aligned}$$

2.5 Optimal Control Using Integrals of Motion

Consideration of control applications for motion integrals was not part of the work undertaken for this thesis, but the potential for such applications *was* a motivation for pursuing the research. One such application was discussed by Berbyuk [19]. The following development is an example of how integrals of motion are used to construct optimal control laws as laid out in [19].

The problem to be solved is as follows: Determine the admissible controllable process $\{\boldsymbol{x}_\star(t), \boldsymbol{u}_\star(\boldsymbol{x}, t)\}$ for the system described by

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t) + B(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}, t), \tag{2.24}$$

that minimizes the functional

$$J[\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)] = \frac{1}{2}\int_0^T \sum_{j=1}^{m} \left[\frac{u_j(x,t)}{k_j}\right]^2 dt, \tag{2.25}$$

on the set Ω, where k_j are prescribed constants. An *admissible controllable process* is a pair $\{\boldsymbol{x}(t), \boldsymbol{u}(t)\}$ satisfying Equation (2.24) and the boundary conditions

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \;\; \boldsymbol{x}(T) = \boldsymbol{x}_T. \tag{2.26}$$

Theorem 1. *Suppose the functional $w(\boldsymbol{x}, t)$ is a first integral of*

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t), \tag{2.27}$$

such that the following conditions hold:

1. *The functional*

$$G[\boldsymbol{x}(\cdot)] = \int_0^T \sum_{j=1}^m k_j^2 \langle \nabla_x w, \boldsymbol{b}_j \rangle^2 dt \tag{2.28}$$

is defined on the entire set Ω*, where* $\boldsymbol{b}_j$ *denotes the* j^{th} *column of* B*.*

2. *The solution of the Cauchy problem,*

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t) + B(\boldsymbol{x})\boldsymbol{u}^0(\boldsymbol{x}, t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2.29}$$

$$u_j^0(\boldsymbol{x}, t) = -k_j^2 \langle \nabla_x w, b_j \rangle,\ j = 1, \ldots, m, \tag{2.30}$$

satisfies $\boldsymbol{x}(T) = \boldsymbol{x}_T$*.*

3. *The following identities hold:*

$$\nabla_{x_i} \left(\sum_{j=1}^m k_j^2 \langle \nabla_x w, b_j \rangle^2 \right) = 0,\ t \in [0, T],\ i = 1, \ldots, n. \tag{2.31}$$

Then the solution is

$$\boldsymbol{x}_\star(t) = \boldsymbol{x}^0(t),\ \ \boldsymbol{u}_\star(x, t) = \boldsymbol{u}^0(\boldsymbol{x}, t),$$

where $\boldsymbol{x}^0$ *and* $\boldsymbol{u}^0$ *are determined by Equations (2.29) and (2.30).*

Proof. Consider the functional

$$\begin{aligned} \Phi[\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)] &= w(x_T, T) + G[\boldsymbol{x}]/2 + J[\boldsymbol{x}, \boldsymbol{u}] \\ &= w(x_T, T) + \frac{1}{2} \int_0^T \sum_{j=1}^m k_j^2 \langle \nabla_x w, \boldsymbol{b}_j \rangle^2 dt + \frac{1}{2} \int_0^T \sum_{j=1}^m \left[\frac{u_j(x, t)}{k_j} \right]^2 dt. \end{aligned}$$

One can see that the extremals of Φ and J are the same on Ω since the functionals

differ by an additive constant. The total derivative of the functional w is

$$\frac{dw(\boldsymbol{x},t)}{dt} = \frac{\partial w}{\partial t} + \langle \nabla_x w, \boldsymbol{f} + B\boldsymbol{u} \rangle.$$

Integrating with respect to time from 0 to T,

$$w(\boldsymbol{x}_T, T) - w(\boldsymbol{x}_0, 0) = w(\boldsymbol{x}_T, T) - w(\boldsymbol{x}_0, 0) + \int_0^T \langle \nabla_x w, \boldsymbol{f} + B\boldsymbol{u} \rangle dt,$$

and since

$$\frac{\partial w}{\partial t} + \langle \nabla_x w, \boldsymbol{f} \rangle = 0,$$

then

$$w(\boldsymbol{x}_T, T) = w(\boldsymbol{x}_0, 0) + \int_0^T \langle \nabla_x w, B\boldsymbol{u} \rangle dt. \tag{2.32}$$

Substituting Equation (2.32), one finds

$$\begin{aligned} \Phi[\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)] &= w(x_T, T) + \frac{1}{2}\int_0^T \sum_{j=1}^m k_j^2 \langle \nabla_x w, \boldsymbol{b}_j \rangle^2 dt + \frac{1}{2}\int_0^T \sum_{j=1}^m \left[\frac{u_j(x,t)}{k_j}\right]^2 dt \\ &= w(\boldsymbol{x}_0, 0) + \frac{1}{2}\int_0^T \sum_{j=1}^m \left[k_j \langle \nabla_x w, \boldsymbol{b}_j \rangle + \frac{u_j(x,t)}{k_j}\right]^2 dt. \end{aligned}$$

It follows that the controllable process defined by Equations (2.29) and (2.30) minimizes the functional $\Phi[\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)]$. □

Example: Consider a rotating rigid body, whose dynamics in the principal frame are given by

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 &= u_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= u_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= u_3, \end{aligned}$$

which, when $\boldsymbol{u} = \mathbf{0}$, has two first integrals:

$$\begin{aligned} \psi_1 &= I_1\omega_1 + I_2\omega_2 + I_3\omega_3 \\ \psi_2 &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2. \end{aligned}$$

Next consider the following transformation of coordinates:

$$x_i = I_i\omega_i,$$

in which case, Euler's equations become

$$\begin{aligned} \dot{x}_1 &= \left(\frac{I_2 - I_3}{I_2 I_3}\right) x_3 x_2 + u_1 \\ \dot{x}_2 &= \left(\frac{I_3 - I_1}{I_1 I_3}\right) x_1 x_3 + u_2 \\ \dot{x}_3 &= \left(\frac{I_1 - I_2}{I_2 I_1}\right) x_2 x_1 + u_3. \end{aligned}$$

In this case

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When $\boldsymbol{u} = \mathbf{0}$,

$$x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 = 0,$$

hence

$$x_1^2 + x_2^2 + x_3^2 = constant.$$

N.B. This constant can also be derived from the fact that in the absence of external torques, the angular momentum vector is an integral of the motion and so is its

norm.

Let

$$\psi(\boldsymbol{x}) \equiv \sqrt{x_1^2 + x_2^2 + x_3^2},$$

and let $w(\boldsymbol{x}) = C\psi$, where C is a constant that has yet to be determined, and for the constants that appear in the cost functional J, $k_1 = k_2 = k_3 = k$ in this case.

$$u_i = -Ck_i^2 \frac{x_i}{\psi}$$

$$\frac{d\psi}{dt} = \nabla_x \psi \cdot (\boldsymbol{f} + \boldsymbol{u}) = \nabla_x \psi \cdot \boldsymbol{u} = -Ck^2,$$

and since

$$\frac{dw}{dt} = \nabla_x w \cdot \boldsymbol{u} = C\frac{d\psi}{dt}$$

$$\nabla_x w \cdot \boldsymbol{u} = -C^2 k^2$$

$$w(\boldsymbol{x}_T, T) = w(\boldsymbol{x}_0, 0) + \int_0^T \nabla_x w \cdot \boldsymbol{u} dt = w(\boldsymbol{x}_0, 0) - C^2 k^2 T$$

$$w(\boldsymbol{x}_T, T) = 0$$

implies

$$C = \frac{\psi(\boldsymbol{x}_0, 0)}{k^2 T}$$

Therefore,

$$u_i(\boldsymbol{x}, t) = -k_i^2 \langle \nabla_x w, b_j \rangle = -x_i \frac{\psi(\boldsymbol{x}_0, 0)}{T\psi(\boldsymbol{x}, t)}$$

is the energy-optimal control.

In the interest of evaluating stability of this control, consider the following Lyapunov function:

$$V = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{x}. \tag{2.33}$$

Taking the first derivative,

$$\begin{aligned}
\dot{V} &= \boldsymbol{x}^T \dot{\boldsymbol{x}} \\
&= \left[\left(\frac{I_2 - I_3}{I_2 I_3} \right) + \left(\frac{I_3 - I_1}{I_1 I_3} \right) + \left(\frac{I_1 - I_2}{I_2 I_1} \right) \right] x_1 x_2 x_3 - \frac{\psi(\boldsymbol{x}_0, 0)}{T\psi(\boldsymbol{x}, t)} (x_1^2 + x_2^2 + x_3^2) \\
&= -\frac{\psi(\boldsymbol{x}_0, 0)}{T\psi(\boldsymbol{x}, t)} (x_1^2 + x_2^2 + x_3^2),
\end{aligned}$$

thus, $\dot{V} \leq 0$ for all $\boldsymbol{x}$. It follows that the control produces asymptotically stable motion.

3. NUMERICAL COMPUTATION OF INTEGRALS

In this chapter we consider a numerical method for computing integrals of motion for nonlinear dynamical systems.

3.1 Motivation

A rotating rigid body is a third-order dynamical system described by a set of nonlinear differential equations,

$$I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\boldsymbol{\omega} = \boldsymbol{\tau}, \tag{3.1}$$

where $\boldsymbol{\omega}$ is the angular velocity vector, $\boldsymbol{\tau}$ represents external torques, and the inertia matrix, I, is constant. If the appropriate coordinate transformations are performed so that the only non-zero components of the inertia matrix are the principal inertias, then this vector equation can be written as three scalar equations,

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1 \tag{3.2}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = \tau_2 \tag{3.3}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = \tau_3. \tag{3.4}$$

This system is known to have two integrals of motion in the absence of external torques (*i.e.*, $\tau_1 = \tau_2 = \tau_3 = 0$): the total energy, and the total angular momentum. Expressed in terms of the principal moments of inertia these are, respectively,

$$\psi_1 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_1\omega_2^2, \tag{3.5}$$

$$\psi_2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_1^2\omega_2^2. \tag{3.6}$$

Evaluating the Wronskian of these two functions, one finds that these two functions are indeed linearly independent over the interval $\boldsymbol{\omega} \in \mathbb{R}^3$.

Recall that for a system described by the differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \tag{3.7}$$

by definition an integral of motion, ψ, satisfies the equation

$$\frac{d\psi}{dt} = \nabla\psi \cdot \boldsymbol{f} + \frac{\partial\psi}{\partial t} = 0. \tag{3.8}$$

If the system is autonomous, as it happens to be for this example, then it is simply

$$\frac{d\psi}{dt} = \nabla\psi \cdot \boldsymbol{f}. \tag{3.9}$$

Now if hypothetically one did not know the forms of the integrals for this system one could make a naïve guess with a power series,

$$\psi = a_1\omega_1^2 + a_2\omega_2^2 + a_3\omega_3^2. \tag{3.10}$$

Subject to the constraint $\nabla\psi \cdot \boldsymbol{f} = 0$, one then has a single equation linear in the coefficients,

$$\nabla\psi \cdot \boldsymbol{f} = a_1\frac{(I_3 - I_2)}{I_1} + a_2\frac{(I_1 - I_2)}{I_2} + a_3\frac{(I_1 - I_2)}{I_3} = 0. \tag{3.11}$$

This problem is underdetermined because there are three unknowns but only one equation. However, if one were to set the value of the one unknown, then the problem could be posed as a constrained minimization problem. That is, taking $a_1 = 1$, one

has

$$a_2\frac{(I_1-I_2)}{I_2}+a_3\frac{(I_1-I_2)}{I_3}=\frac{(I_2-I_3)}{I_1},$$

which can be written in the form

$$A\boldsymbol{x}=\boldsymbol{c},$$

where $\boldsymbol{x}=[a_2\,a_3]^T$. Minimization of a quadratic subject to a constraint $A\boldsymbol{x}=\boldsymbol{c}$ yields the solution

$$\boldsymbol{x}=A^T(AA^T)^{-1}\boldsymbol{c}+[A^T(AA^T)^{-1}A-I]\boldsymbol{z}, \tag{3.12}$$

where $\boldsymbol{z}$ is an arbitrary vector. Time spent playing with the values for z leads one to discover that the known integrals of motion can be recovered when

$$z_1=z_2=-2.75-5k,\quad k\in\mathbb{Z},$$

from which it follows that

$$a_j=I_j(kI_j-k+1),$$

The two known conserved quantities of this system are recovered when $k=0$ and $k=1$,

$$\begin{aligned} k=0: &\qquad \psi=I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2=2E\\ k=1: &\qquad \psi=I_1^2\omega_1^2+I_2^2\omega_2^2+I_3^2\omega_3^2=h^2. \end{aligned}$$

In the absence of a kinematic relation for the angular coordinates this system should possess only two integrals of motion. Indeed, it appears that one cannot produce

any more than two independent integrals with this method. For example, for $k = 2$,

$$\psi = (2I_1^2 - I_1)\omega_1^2 + (2I_2^2 - I_2)\omega_2^2 + (2I_3^2 - I_3)\omega_3^2 = 2h^2 - 2E.$$

The rest of this chapter discusses a method that, although technically different, is based on the same premise.

3.2 The SVD Method

As a generalization of the method discussed in the last section, consider the *ansatz*

$$\psi(\boldsymbol{x}) = \sum_n a_n F_n(\boldsymbol{x}) \tag{3.13}$$

for a candidate motion integral of the dynamical system described by the equations of motion

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}). \tag{3.14}$$

By definition, the candidate integral is required to satisfy

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial \boldsymbol{x}} \cdot \dot{\boldsymbol{x}} = \frac{\partial \psi}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x}) = 0. \tag{3.15}$$

Substituting the integral *ansatz*, Equation (3.13), one has

$$\sum_n a_n \nabla F_n(\boldsymbol{x}) \cdot \boldsymbol{f} = 0, \tag{3.16}$$

which can be re-expressed in matrix form as

$$(\boldsymbol{f} \cdot \nabla F)^T \boldsymbol{a} = 0. \tag{3.17}$$

Generally, the functions F_n can be any set of polynomials in the state space variables, but some dynamical systems might benefit from a set of polynomials that respects its symmetries. This particular issue will be discussed later in the thesis.

Equation (3.17) describes a system of one equation for n variables. Of course, as long as the number of unknowns is greater than one, one will always need more than one equation to solve the system. The method described here entails constructing a matrix of full rank or nearly full rank from the row vector $(\boldsymbol{f} \cdot \nabla F)^T$ by sampling the phase space to populate the matrix rows. The problem of solving for the coefficients $\boldsymbol{a}$ is then recast into the familiar problem of determining the kernel (or nullspace) of a matrix. This is a problem to which the singular value decomposition (SVD) is well-suited. The matrix whose rows are given by $(\boldsymbol{f} \cdot \nabla F)^T(\boldsymbol{x})$, is hereafter referred to as the *integrability condition matrix.*

Singular value decomposition is a matrix factorization of the form

$$A = USV^*,$$

where U and V are unitary matrices, and S is a diagonal matrix whose entries are the singular values. The columns of V and, independently the rows of U, form an orthonormal basis of the null space. A *singular value* of the matrix A is a non-negative real number σ such that

$$A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$$

and

$$A^*\boldsymbol{u}_i = \sigma_i \boldsymbol{v}_i,$$

where the left- and right-singular vectors vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$, are the columns of U and rows of V, respectively, corresponding to the singular value σ_i. In the context

of the problem discussed here, namely, that of determining the nullspace of the matrix $(\boldsymbol{f} \cdot \nabla F)^T$, the left-singular vectors corresponding to vanishing singular values constitute a basis for the nullspace, and hence, a solution for the vector $\boldsymbol{a}$ in Equation (3.17).

This is the basis for the SVD method of numerically computing integrals. The form of the *ansatz* and details of the sampling method are covered by the implementation .

3.3 Implementation

There are several variables that can affect the performance:

1. Choice of *ansatz*.
2. Length of *ansatz*.
3. Sampling method.
4. Bounds of sample space.
5. Number of sample points.

3.3.1 The Ansatz

The choice of a regular polynomial for the candidate integral is the simplest, but it is not always the best. For instance, a truncated regular polynomial may not respect the system's symmetries. In such a case, a candidate integral comprised of orthogonal polynomials might better approximate the phase space geometry.

As for series expansions of any function, we expect that the more terms included in the *ansatz*, the better the agreement with the true integral. When linearizing a system of differential questions in order to obtain a solution, one operates with the understanding that any resulting solution is only valid as long as the neglected

nonlinear terms are not significant. Effectively, the dynamics for a linearized systems are restricted to a flat sub-region of the phase space.

With regard to the analytic form of the motion integrals, linearization of the nonlinear system differential equations produces the same result as truncation of the integral of the full nonlinear system to first-order. For example, if one were to linearize the motion integral of the simple pendulum one would end up with that of the harmonic oscillator. Truncating the polynomial *ansatz* is a step beyond linearization in the sense that some of the nonlinear behavior is accounted for, but not all of it. The effect of truncating terms from the motion integral *ansatz* is that the function may not capture the geometry with fidelity as one wanders from an equilibrium point in phase space.

3.3.2 Sampling

The method considered here samples the phase space in order to generate enough equations to make a solvable system. Proceeding in this way, it is assumed that Equation (3.17) holds point by point throughout the phase space. This assumption is used to generate a system of equations by sampling the phase space. With in infinity of points to choose from, one can construct a well-determined system or an over-determined system, though some care must be exercised when choosing how to sample to avoid producing a rank deficient matrix.

There are a number of conceivable ways of sampling, but here we consider only three: grid, trajectory-based grid and trajectory-based random sampling. Grid sampling takes points in phase space within a given set of bounds. There are two different types of trajectory-based sampling. Both entail numerical solution of the system prior to solution of the *ansatz* coefficients. The simplest method uses points along the solution trajectory in phase space to populate the rows of the integrability con-

dition matrix. The other method first chooses from a uniform distribution a set of initial conditions within a given radius of some given point in phase space. Then the system is numerically integrated for each of the initial conditions. The sample points are taken from each trajectory to populate the integrability condition matrix.

The method's effectiveness will also be dependent upon the sample space bounds. An improperly sized sample space can produce erroneous results in two ways. A sample space that is too small to capture higher-order nonlinear behavior can result in a candidate integral incapable of following dynamics beyond the sample space bounds. Similarly, a large sample space for an integral with too few terms can produce erroneous results because the method will attempt to fit a weakly nonlinear, or even linear, integral to a highly nonlinear phase space, for example. In general, the larger the sample space, the more terms should be included in the candidate integral *ansatz*.

3.3.3 Note on Testing & Validation

We have chosen two dynamical systems on which to test this technique. The simple pendulum is a second-order nonlinear system possessing one integral of motion, which can be analytically derived. The rotating rigid body is a third-order nonlinear system that possesses two integral of motion, which can also be analytically derived. These two examples were chosen to validate the technique since in both cases the exact integrals of motion are known. The second example was chosen to test ideas regarding linear independence of integrals, which cannot be tested for the case of the simple pendulum since the system possesses only a single first integral. Moreover, the rotating rigid body has widely been used as a physical model for problems in dynamics and control in aerospace engineering.

Validation is done by first numerically integrating to determine a solution for a

given set of initial conditions. The resulting solution is then feed into the candidate integral, which is then plotted with respect to time. Some variation is expected due to integration error and truncation error, but generally the magnitude of the error should be controllable via the integration error tolerances and the number of terms included in the *ansatz*.

The quality of fit is evaluated primarily by the error between the values of the approximate and true integrals along a prescribed trajectory. The state space plots are also compared to evaluate the extent to which the quality of fit extends beyond the sampled subspace. The error is given by

$$\text{error in } \psi \equiv \overline{\psi} - \overline{\psi}_{true}, \tag{3.18}$$

where the rescaled integrals $\overline{\psi}$ and $\overline{\psi}_{true}$ are used to deal with the constant offset between the true and candidate integrals. The rescaled integrals are given by

$$\overline{\psi}(t_i) \equiv \psi(t_i) - \frac{1}{N}\sum_j \psi(t_j) \tag{3.19}$$

$$\overline{\psi}_{true}(t_i) \equiv \psi_{true}(t_i) - \frac{1}{N}\sum_j \psi_{true}(t_j), \tag{3.20}$$

where N is the number of data points taken in the course of the simulation.

The error surfaces are given by $|\overline{\psi}_{true} - \overline{\psi}|$. In order to make a meaningful comparison, both the true and candidate integral surfaces are shifted along the z-axis prior to computing the difference so that they have the same value at the origin.

3.4 A Simple One-Dimensional Nonlinear System

The simple pendulum is a second-order nonlinear system described by the differential equation

$$\ddot{\theta} + \omega^2 \sin\theta = 0, \tag{3.21}$$

where $\omega \equiv \sqrt{g/l}$, g is the gravitational force at the surface of the Earth, and l is the length of the pendulum. This system has one integral which can be analytically derived in the following way.

For a dynamical system described in terms of state space variables by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \tag{3.22}$$

an integral of motion, ψ, must satisfy the differential equation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial \boldsymbol{x}} \cdot \dot{\boldsymbol{x}} = \frac{\partial\psi}{\partial \boldsymbol{x}} \cdot \boldsymbol{f} = 0. \tag{3.23}$$

The simple pendulum is described by the state space equations

$$\boldsymbol{x} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}, \quad \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ -\omega^2 \sin\theta \end{pmatrix}. \tag{3.24}$$

Then the differential equation for the first integral is

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\dot{\theta}}\ddot{\theta} = \frac{\partial\psi}{\partial\theta}\dot{\theta} - \omega^2 \frac{\partial\psi}{\partial\dot{\theta}} \sin\theta = 0. \tag{3.25}$$

One can then solve by separation of variables in the following way: make the *ansatz*

$$\psi(\theta, \dot{\theta}) = \psi_1(\theta) + \psi_2(\dot{\theta}), \tag{3.26}$$

so one now has

$$\frac{d\psi_1}{d\theta}\dot{\theta} - \omega^2 \frac{d\psi_2}{d\dot{\theta}} \sin\theta = 0, \tag{3.27}$$

which implies that

$$\frac{d\psi_1}{d\theta} = C\omega^2 \sin\theta \tag{3.28}$$

$$\frac{d\psi_2}{d\dot{\theta}} = C\dot{\theta}, \tag{3.29}$$

where C is a constant. Integrating produces the solution

$$\psi = \frac{1}{2}\dot{\theta}^2 - \omega^2 \cos\theta + \beta, \tag{3.30}$$

where β is a constant of integration. When $\beta = \omega^2$, ψ represents the total energy of the system.

3.4.1 Degree for a Regular Polynomial

The following scenarios were setup so that the higher-order terms made a significant contribution. To that end, the initial conditions used in the simulation were $\theta_0 = \pi/2$, $\dot{\theta}_0 = 0$. Sample points were taken from a grid with bounds $\theta \in [-\pi/2, \pi/2]$ and $\dot{\theta} \in [-\pi/2, \pi/2]$. For all examples of the simple pendulum, the frequency is $\omega = 1$. The polynomial has the general form

$$\psi = a_1\theta + a_2\theta^2 + \cdots + b_1\dot{\theta} + b_2\dot{\theta}^2 + \cdots$$

The plots of the candidate integrals tend to oscillate simply because the simple pendulum is generally an oscillatory system, but the motion of the error in ψ is mostly irrelevant. The scale of the maxima and minima serve as an indicator of how well the candidate integral approximates the true integral. Figures A.1, A.2, and A.3

plot the difference between candidate and true integrals over time. Apart from the high frequency behavior due to the oscillatory nature of the system, this plot should not exhibit any long wavelength behavior as the simple pendulum is a stationary system. And indeed, long time-scale error appears to be flat.

These plots of the error over time show how well the candidate integral approximates the true integral for one trajectory with a single set of initial conditions. In particular, they show that as the degree of the polynomial increases, the candidate integral better approximates the true integral. These results are in line with the expectation that the degree of the polynomial *ansatz* increases, so should agreement between the integrals.

The phase space error plots provide an indication of the fidelity of the candidate integral over a larger portion of the phase space, as opposed to just over a single trajectory. These two-dimensional[1] plots show where in phase space the candidate and true integrals disagree with each other, and by how much. The integrals of motion define the phase space geometry. With higher-order terms, the candidate integral *ansatz* can better approximate the phase space geometry further away from the origin. So it is natural to expect that as higher-order terms are included, one should observe flattening phase space error plots.

The phase space error plots for this first test case are shown in Figures A.4, A.5, and A.6. We know that the true integral for this system contains a cosine, which can be represented as an infinite even-degree Taylor series. So we are justified in expecting these plots to indicate better agreement between the candidate and true integrals as the degree of the candidate integral polynomial increases. And indeed, we are not disappointed as the plots show that as the order of polynomial *ansatz*

[1]Of course, the phase space is only two-dimensional for systems with two degrees of freedom. Similar plots for larger system would be "slices" of the phase space.

increases, the candidate integral better approximates the true integral further from the origin.

To test the effectiveness of the candidate integral for other sets of initial conditions, comparisons done with true integrals computed for initial conditions closer to the stable equilibrium were made. We expect that for trajectories with initial conditions within the bounds of the sample space or close to the equilibrium, the integral generated from this method should remain reasonably constant. We also expect that for trajectories with initial conditions further away from the equilibrium point, the candidate integral should be less constant, which would be indicated by a larger time scale on an integral error vs. time plot. The results of those simulations are shown in Figures A.7 and A.8. One can observe the agreement between the candidate integral and true integral improves as the initial conditions approach the stable equilibrium.

3.4.2 Testing Fourier and Mixed-Fourier Expansions

The problem with the polynomial series used thus far is that while the cosine function can be represented as a series expansion in the variable θ, that series is infinite. Knowing the true integral of motion for this system provides a certain advantage when posing a candidate integral. We know that wherever one decides to truncate the series for the candidate integral, the cosine term is merely being approximated.

For the simple pendulum, given that the equations of motion can be expressed in terms of sines and cosines, a more judicious choice might be $\cos(n\theta)$ or $\sin(n\theta)$. In fact, aside from the term quadratic in $\dot{\theta}$, one might expect that with an expansion in cosine the SVD method could pick out the second term in ψ that we know to be there. We can consider a full Fourier-type *ansatz* with both sines and cosines for all

variables,

$$\psi(\theta, \dot{\theta}) = \sum_n \left[\sin(n\theta) + \cos(n\theta) + \sin(n\dot{\theta}) + \cos(n\dot{\theta}) \right]. \tag{3.31}$$

But we can go further. While the pure Fourier-type *ansatz* may capture the cosine term we know to be there, the term quadratic in $\dot{\theta}$ must be culled from amongst the even-degree terms in a the cosine terms of that variable. One might consider a mixed Fourier form for the candidate integral wherein a simple polynomial in the variables is mixed with a truncated Fourier series,

$$\psi = \left[\sum_n (\theta^n + \dot{\theta}^n) \right] \left[\sum_m (\sin(m\theta) + \cos(m\theta) + \sin(m\dot{\theta}) + \cos(m\dot{\theta})) \right]. \tag{3.32}$$

This candidate integral contains everything necessary to match the true integral. It would then just be up to the method to pick out the right terms.

The results for both candidate integrals are given in Figures A.9 through A.16. Analysis of these results is left to Section 3.4.5.

3.4.3 Testing Sample Method: Trajectory

The idea behind the trajectory sampling method is that if there is anywhere in phase space that lies on an integral surface it will be along a trajectory of the system. Using that assertion, the system in question is numerically integrated given some initial conditions, and points in that solution are used as sample points with which to solve for the unknown coefficients of the integral *ansatz*.

Initial tests were done using the same parameters as previous tests. The trajectory from which the sample points was taken was also that used to compare the candidate integral with the true integral. The results follow in Figures A.17 and A.18, which indicate slightly poorer performance as compared to that of the gridded sampling method. This result seems to conflict with the expectation that the fit should have

been much better, especially considering that the sample points were chosen from a real system trajectory.

A potential detractor of this method is that the resulting integral could fail to be an integral for other trajectories. That concern was addressed by comparing the candidate integral with the true one along a trajectory different from that used for the sample points. As Figure A.19 indicates, this concern is well-founded when considering initial conditions further from stable equilibrium. However, the fit is better for initial conditions closer to the stable equilibrium, as indicated in Figure A.20 .

3.4.4 Testing Sample Method: Trajectory Bundle

The trajectory bundle method extends the previously used trajectory method by adding a random element to the sample point selection. An initial condition is given as the center point of a ball in phase space. Then for a set of uniformly distributed initial condition points within a given radius of center point, each is propagated numerically to create a "bundle" of trajectories.

There are several variables that can impact the effectiveness of the method: initial point center, radius, and number of samples per trajectory. When the integrability condition matrix is required to be well-defined (*i.e.*, full rank), the number of sample points taken from each trajectory determines the number of trajectories in the bundle.

As can be seen in Figures A.21 and A.22, a higher-degree polynomial *ansatz* provides only a marginal improvement. Due to the random nature of the sample point selection, over a number of simulations agreement between the candidate integral and true integral was occasionally seen to improve. There is also little visible improvement for a change in initial point radius as indicated by Figures A.22 and A.23.

When the number of samples per trajectory is increased, the number of trajectories is decreased. So more weight is effectively put on fewer trajectories. Figure A.24 indicates that the number of samples per trajectory has a very weak influence on candidate integral fit.

3.4.5 Analysis

In general, random sampling methods for this application have not performed well, perhaps due to relatively small number of points. That is, for a well-defined system, the number of sample points is determined by the number of terms in the polynomial *ansatz*. Even though the samples are taken as a uniform distribution, it is not unlikely that clustering occurs for a small number of points. Consequently, the integral condition matrix could be less than full rank. In the course of testing we have noticed as many wild improvements as there were degradations with each run, which lends credence to this explanation.

Gridded sampling, on the other hand, would guarantee coverage of the state space with a minimal set of points. One flaw of gridded sampling could be the fact that samples are being taken from parts of the phase space that the system may never visit. In the simulations, the bounds were set to include the volume enveloping the system trajectories for a given set of initial conditions and parameters, but any system exhibiting regular motion will surely only occupy a sub-space within that volume.

It was thought that the trajectory bundle method might prove to be robust for cases where the the system would tend to demonstrate chaotic motion; that is, systems where trajectories with similar initial conditions that diverge. It appears, however, that the method is not as effective as the other methods, even for systems exhibiting regular motion. One potential source of error could be related to the fact

that the initial condition points are sampled randomly, which as noted above, could occasionally lower the rank of the integral condition matrix when clustering occurs.

Comparison of the candidate integrals for different sets of initial conditions, as in Figures A.19 and A.20, supports the validity of the method. That the candidate integrals better approximate the true integrals as the system executes motion closer to the linear regime seems to indicate that the candidate integrals faithfully extend the integrals of motion for the linear system beyond the linear regime.

Phase space error plots for the simple pendulum are given in Figures A.5 and A.6 and for the Fourier and Fourier-mixed methods in Figures A.11, A.12, A.15, and A.16. These plots indicate how well the candidate integral fits over a region of the phase space, not just over one particular trajectory. As noted above, the plots are generated so that the difference between the candidate and true integral is zero at the origin.

Observations from these plots are in line with the previous observation that closer to the region about the origin where the dynamics are essentially linear, all candidate integrals better approximate the true integral. Also, one should expect that the discrepancy between the candidate and true integrals would grow near the margins of sampled phase space, and indeed, that expectation is borne out in these results.

The pure Fourier series *ansatz* does not perform much better than the power series (if not worse). This poor performance may be due to its inability to account for the term quadratic in $\dot{\theta}$. The mixed Fourier *ansatz* should address that deficiency, and indeed it does perform much better. It performs better than its counterpart and much better than the power series *ansatz*. The results, however, are not without curiosities.

The mixed Fourier candidate integral truncated above 2nd order performs noticeably worse than the same candidate integral truncated above the 4th order. The true

integral contains only one term in $\dot{\theta}$, which is quadratic, and only one "1st order" cosine, $\cos\theta$, so it comes as a surprise that an *ansatz* including higher orders of $\dot{\theta}$ and $\cos\theta$ are required for a better fit. Of course, we have not in any way utilized orthogonality of the functions used in expanding the candidate integral, so some imprecision of the fit does not come entirely as a surprise.

3.5 The Rotating Rigid Body

A rotating rigid body is a 3rd-order dynamical system described by the differential equations

$$I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\boldsymbol{\omega} = \boldsymbol{\tau}, \tag{3.33}$$

where $\boldsymbol{\omega}$ is the angular velocity vector, $\boldsymbol{\tau}$ represents external torques, and the inertia matrix, I, is constant. If the appropriate coordinate transformations are performed so that the only non-zero components of the inertia matrix are the principal inertias, then this vector equation can be written as three scalar equations,

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1 \tag{3.34}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = \tau_2 \tag{3.35}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = \tau_3. \tag{3.36}$$

This system differs from the simple pendulum previously considered in several ways: it is a higher-dimensional system and as such, possesses multiple integrals of motion, and the integrals are closed-form polynomials. That is, they can be expressed as a finite number of monomials.

This system is known to have two integrals of motion in the absence of external torques (*i.e.*, $\tau_1 = \tau_2 = \tau_3 = 0$): the total energy, and the total angular momentum.

Expressed in terms of the principal moments of inertia these are, respectively,

$$\psi_1 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_1\omega_2^2, \tag{3.37}$$

$$\psi_2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_1^2\omega_2^2. \tag{3.38}$$

Evaluating the Wronskian of these two functions, one finds that these two functions are indeed linearly independent over the interval $\boldsymbol{\omega} \in \mathbb{R}^3$.

The entire state of a rotating rigid body, however, comprises a set of of three independent angular variables as well as the angular velocities. The differential equations governing the time evolution of these angular variables are kinematic relations between them and the angular velocities.

3.5.1 Analysis

The system model used in these tests is a freely rotating rigid body (read: no external torques). The sample space bounds are $\omega_i \in [-2, 2]$ for all angular velocities. Points in phase space were sampled in a grid so as to produce a full-rank integrability condition matrix, but an over-determined system. Unlike the simple pendulum, the over-determined system produces better results, as can be seen when comparing Figures A.26 and A.25.

Figure A.26 shows the error for the motion integral corresponding to the most singular value with respect to the known motion integrals of this system. The error for the second integral found via the SVD method is shown in Figure A.27. By the nature of the singular value decomposition, the vectors whose values are the coefficients of the candidate integral are orthogonal. The *ansatz* can be thought of as a vector that lives in a space whose basis vectors are its monomials. Then it stands to reason that orthogonality of the coefficient vectors produced by the singular value

decomposition indicates that they are linearly independent.

When compared to Figure A.26, Figure A.28 indicates that in this case, there is effectively no advantage with regard to error in making a higher-order polynomial *ansatz*. This result what we would have expect knowing that the true integrals of this system are both finite polynomials.

4. CONCLUSION

This thesis has considered a numerical method for finding integrals of motion in a systematic way. There is no truly general method for determining whether a system is integrable. Noether's method is the most well-known and widely used method for obtaining motion integrals, but only for Lagrangian systems, and it is capable of only revealing integrals connected with Lagrangian symmetries.

Although the method considered here was not applied to any non-Lagrangian with resounding success, the premise does not explicitly depend upon whether the system is Lagrangian or not. This investigation was undertaken to map out its limitations and to generally elucidate more about the subject of motion integrals with the understanding that the method would not generally be applicable.

The method assumes the existence of an integral with a polynomial form, and its vanishing time derivative is used to solve for the unknown coefficients. The problem is posed as that of determine the nullspace of a matrix whose entries are the terms of the integral *ansatz* evaluated at different points in the phase space. Singular value decomposition is then used to determine the basis vectors of the nullspace, which in this case are the unknown coefficients.

The benchmark systems – the simple pendulum and freely rotating rigid body – provided a means to investigate the effects of a number of variables such as polynomial degree, number of sample points and sample method. Application of this method to systems exhibiting interesting and complicated dynamics should be undertaken in order to better understand the method's limitations. The method as applied here was done so rather naïvely in that no attention was paid to radius of convergence and, in the case of the Lorenz system, whether the dynamics were

chaotic. Further investigation should give careful consideration to these issues to better inform improvements in the method and to better understand its limitations. There is also some more work to be done on sampling methods.

The gridded sampling method was clearly the more effective of the three, but potential flaws have been pointed out. In particular, whether the grid sampling method over-constrains the system should be investigated. It is suspected that the mismatch of phase space plots is evidence of this. Also, more sophisticated methods of sampling the phase space could be considered. The trajectory bundle method was conceived of as an innovation on the brute force grid sampling method, but its performance did not match that of the latter. One of the causes for the mediocre performance of the trajectory bundle method may be the fact that the initial condition points were randomly sampled. As noted in the analysis, the random sampling methods may suffer from a paucity of points. A gridded method, on the other hand, could provide for a wider diversity of points without clustering.

With regard to applications of motion integrals, not much beyond the work of Berbyuk [18, 19] was discussed. Indeed, it seems that optimal control could potentially be a fertile ground for the use of motion integrals. We suggest that standard methods in optimal control for turning motion integrals into control laws such as that in [19], and widely applicable methods for determining motion integrals such as that considered here, could be applied to systems that are known to be non-chaotic and for which an algebraic form of the integrals is yet unknown.

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APPENDIX A

FIGURES

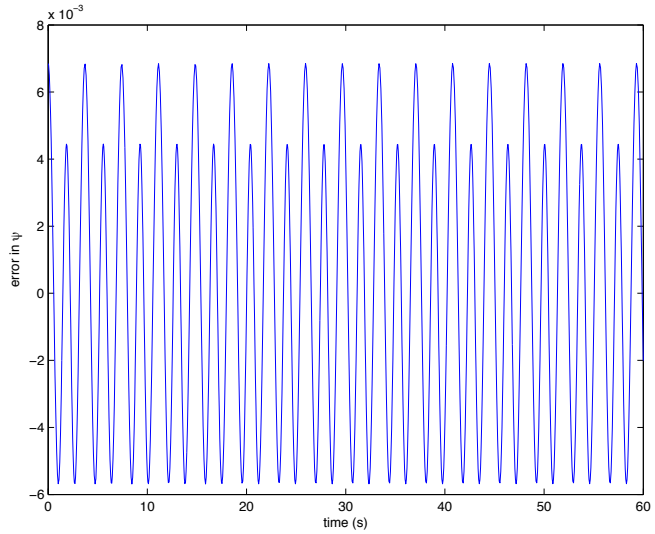

Figure A.1: Error in ψ; sampling method: grid; well-determined; 4th order polynomial.

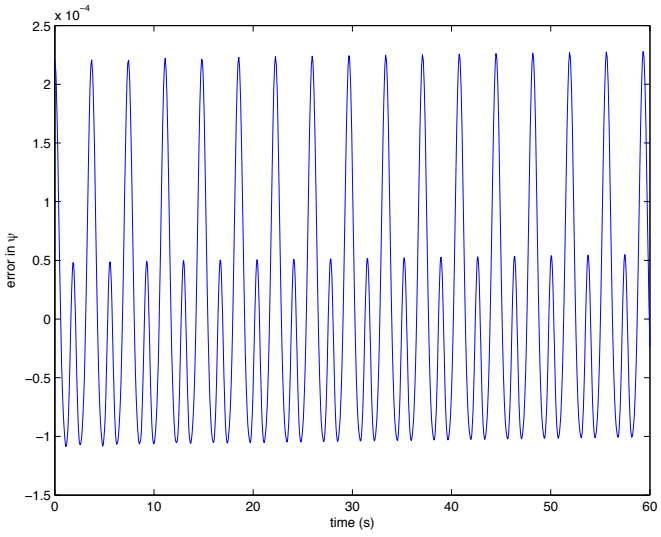

Figure A.2: Error in ψ; sampling method: grid; well-determined; 8th order polynomial.

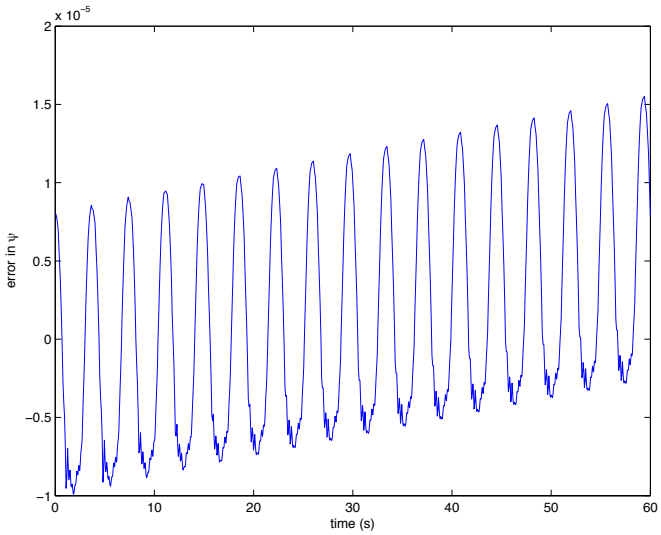

Figure A.3: Error in ψ; sampling method: grid; well-determined; 12th order polynomial.

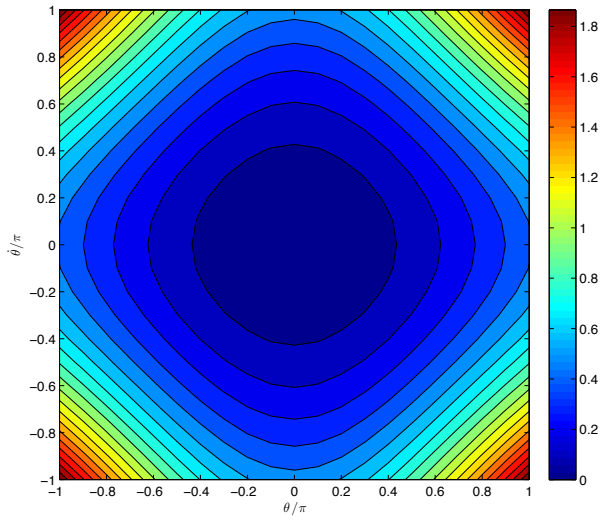

Figure A.4: Error surface; sampling method: grid; well-determined; 4th order polynomial.

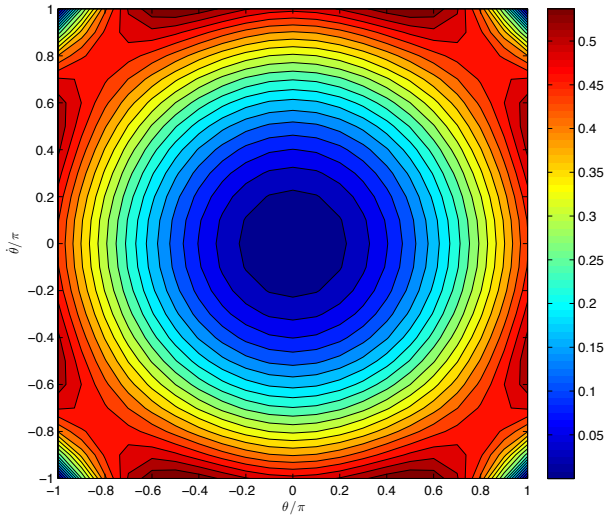

Figure A.5: Error surface; sampling method: grid; well-determined; 8th order polynomial.

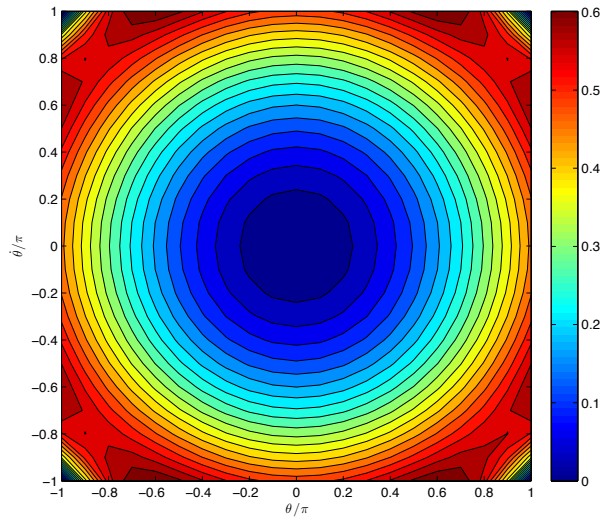

Figure A.6: Error surface; sampling method: grid; well-determined; 12th order polynomial.

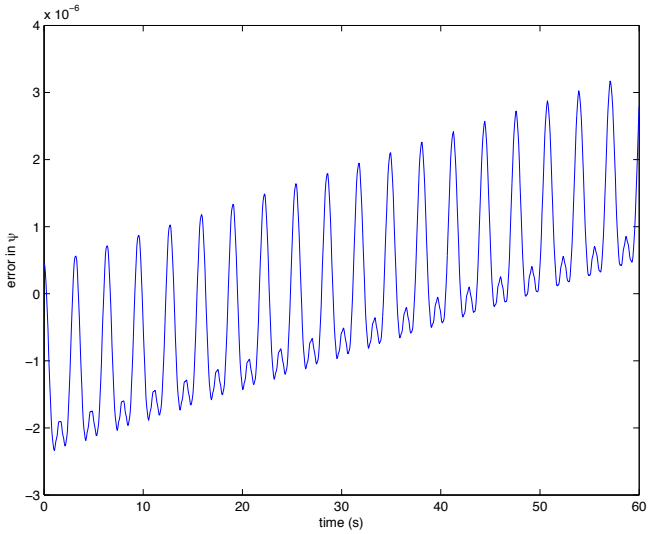

Figure A.7: Error in ψ; sampling method: grid; well-determined; 4th order polynomial; $\theta_0 = \pi/8$.

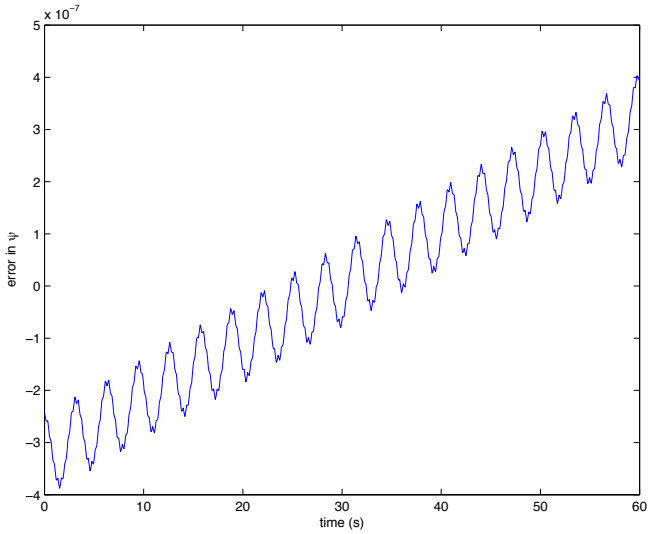

Figure A.8: Error in ψ; sampling method: grid; well-determined; 4th order polynomial; $\theta_0 = \pi/32$.

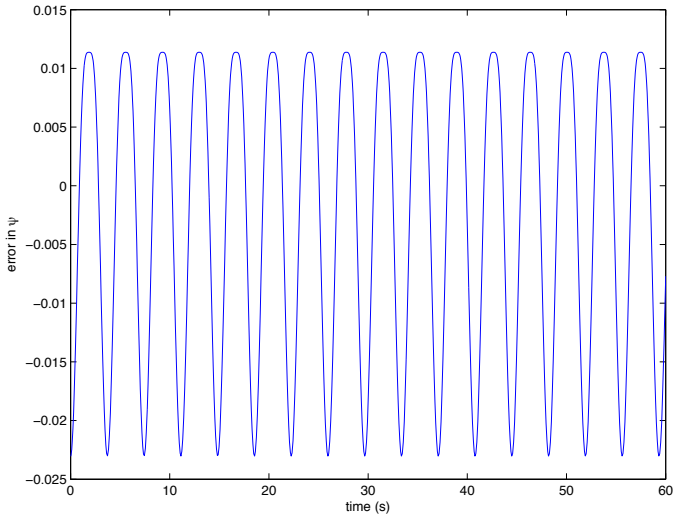

Figure A.9: Error in ψ; grid sampling method; Fourier series: 1st order.

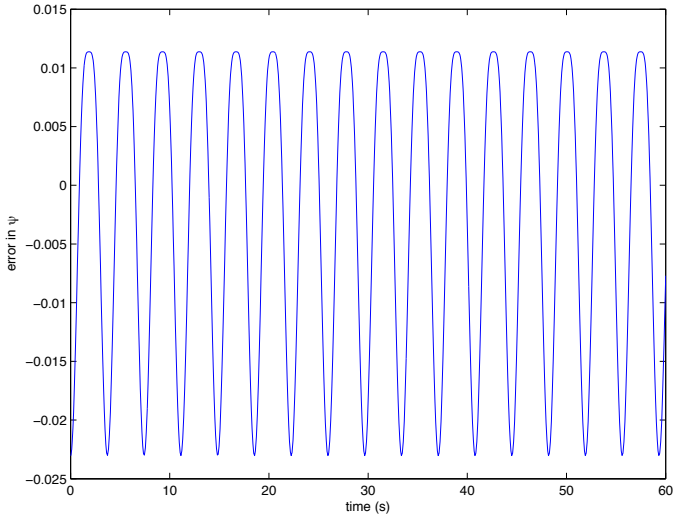

Figure A.10: Error in ψ; grid sampling method; Fourier series: 4th order.

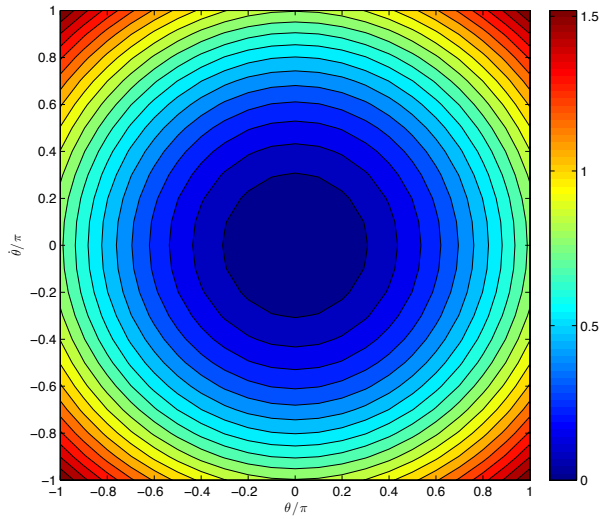

Figure A.11: Error surface; grid sampling method; Fourier series: 1st order.

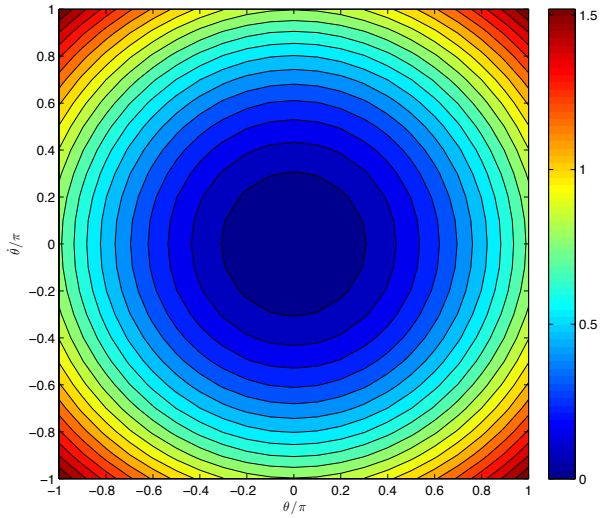

Figure A.12: Error surface; grid sampling method; Fourier series: 4th order.

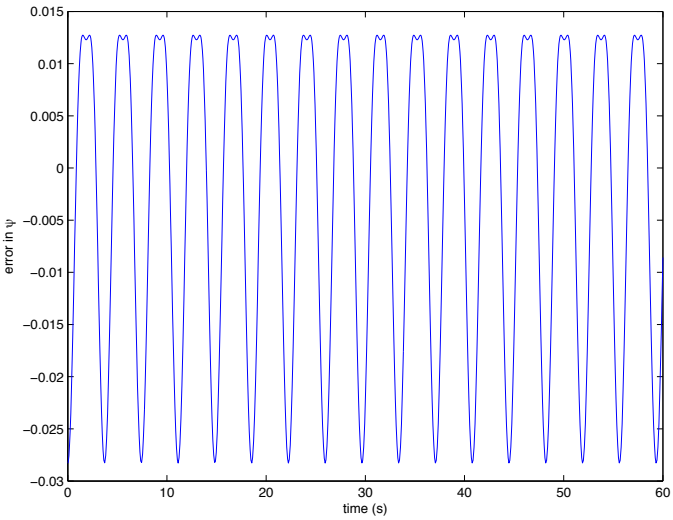

Figure A.13: Error in ψ; grid sampling method; Mixed Fourier series: 2nd order.

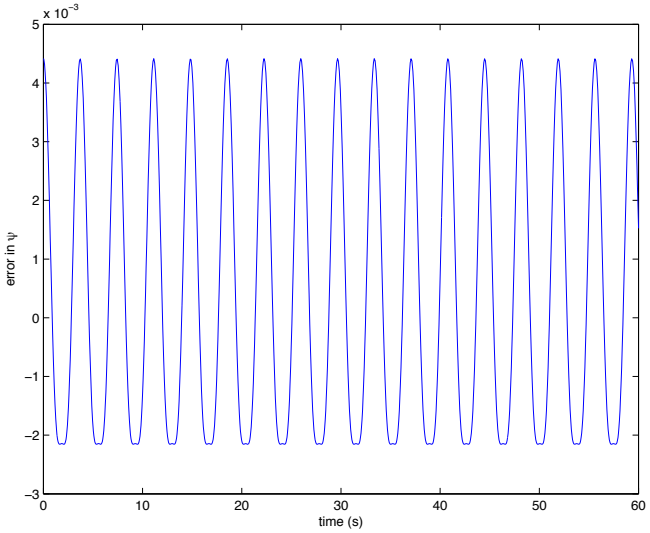

Figure A.14: Error in ψ; grid sampling method; Mixed Fourier series: 4th order.

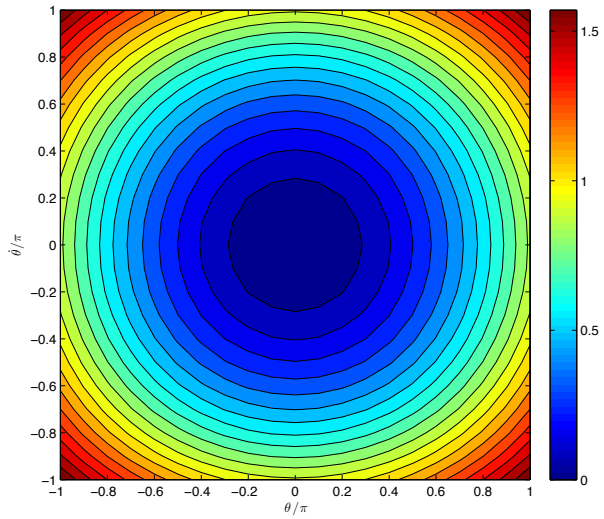

Figure A.15: Error surface; grid sampling method; Mixed Fourier series: 2nd order.

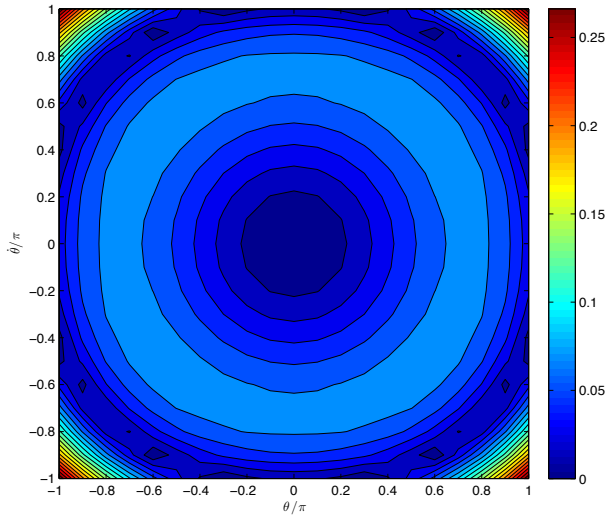

Figure A.16: Error surface; grid sampling method; Mixed Fourier series: 4th order.

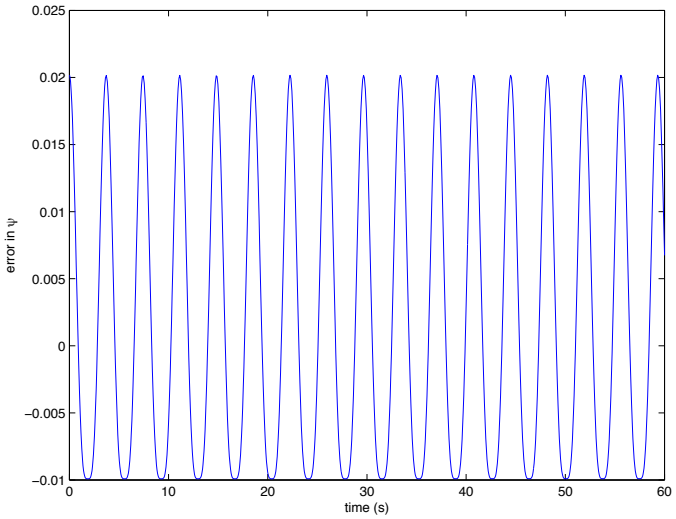

Figure A.17: Sampling method: trajectory; well-determined; 8th order polynomial.

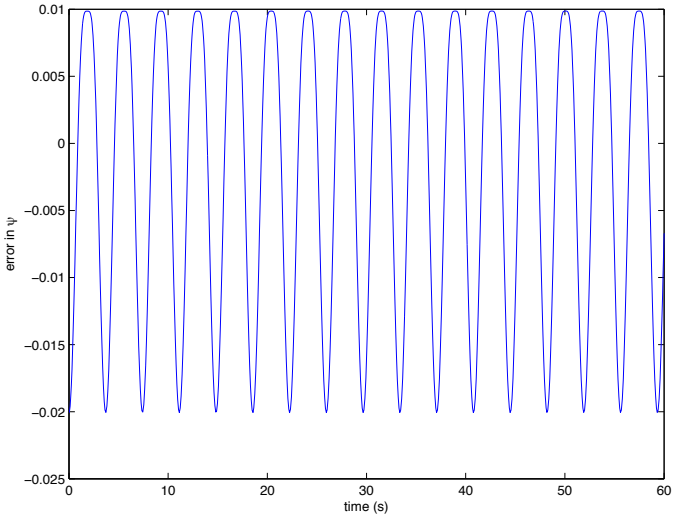

Figure A.18: Sampling method: trajectory; well-determined; 12th order polynomial.

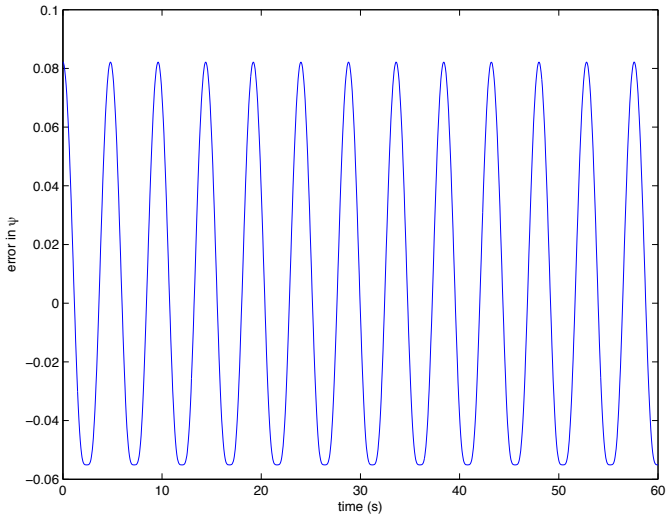

Figure A.19: Sampling method: trajectory; well-determined; 8th order polynomial; $\theta_0 = 3\pi/4$.

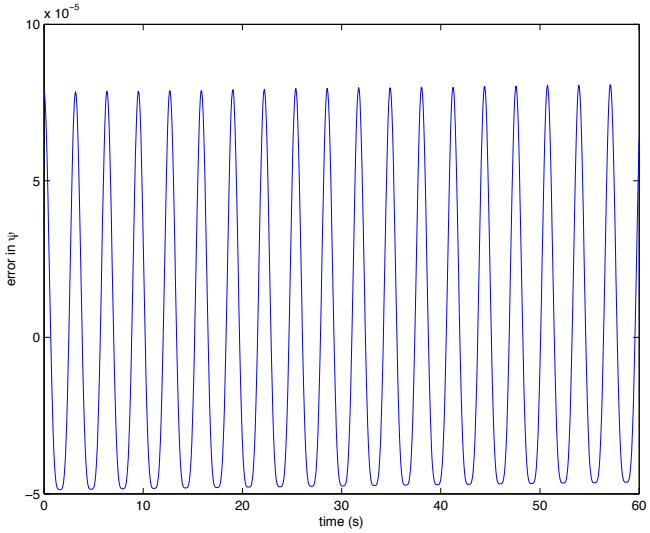

Figure A.20: Sampling method: trajectory; well-determined; 8th order polynomial; $\theta_0 = \pi/8$.

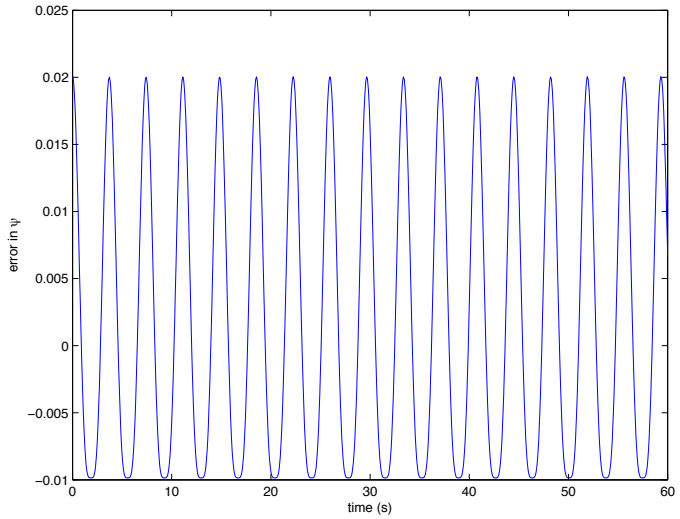

Figure A.21: Sampling method: trajectory bundle; well-determined; 8th order polynomial; radius: 1; samples/trajectory: 3.

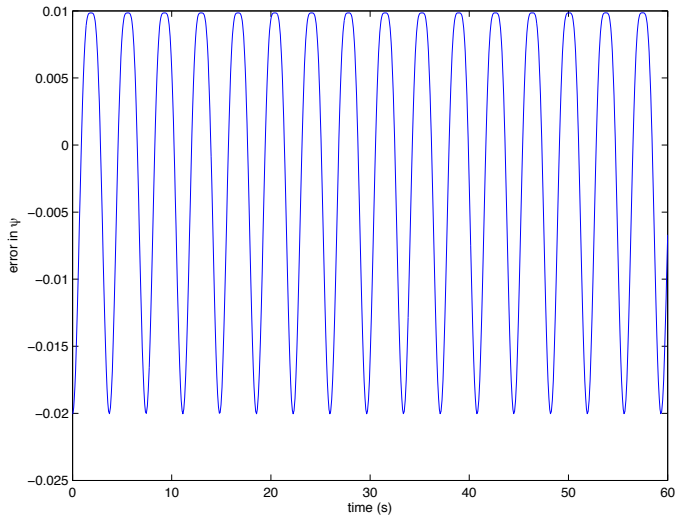

Figure A.22: Sampling method: trajectory bundle; well-determined; 12th order polynomial; radius: 1; samples/trajectory: 3

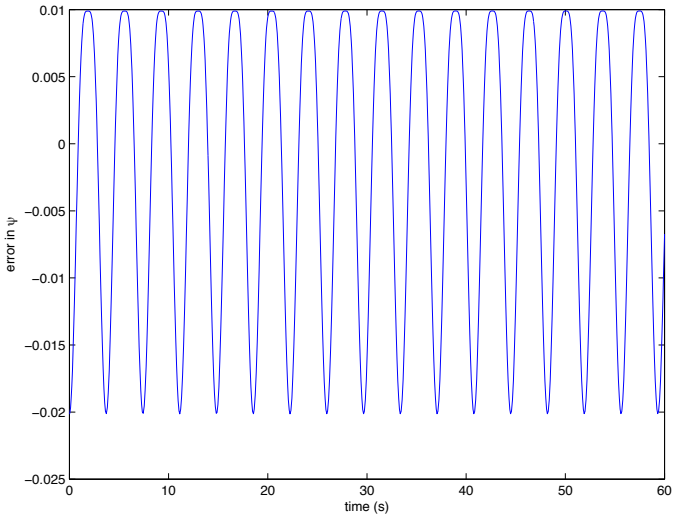

Figure A.23: Sampling method: trajectory bundle; well-determined; 8th order polynomial; radius: 2; samples/trajectory: 3

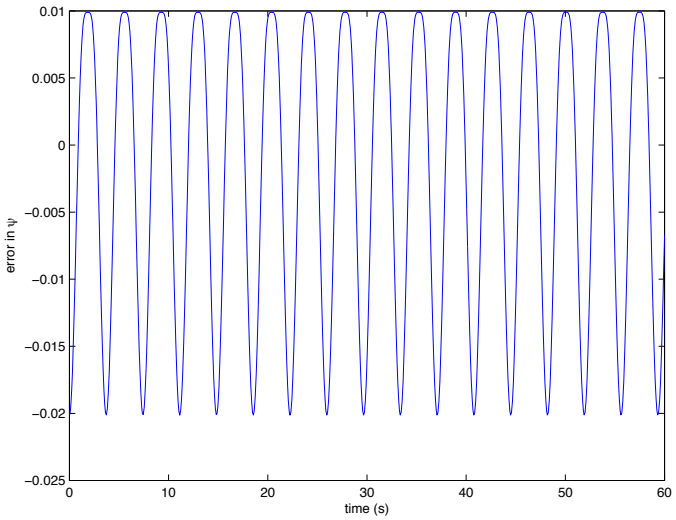

Figure A.24: Sampling method: trajectory bundle; well-determined; 8th order polynomial; radius: 1; samples/trajectory: 4

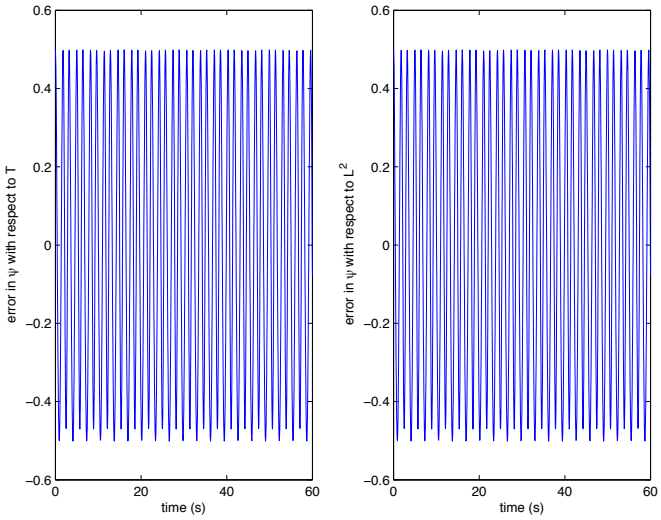

Figure A.25: Freely rotating rigid body; grid sampling method; 6th order polynomial; 1st motion integral; nearly well-determined.

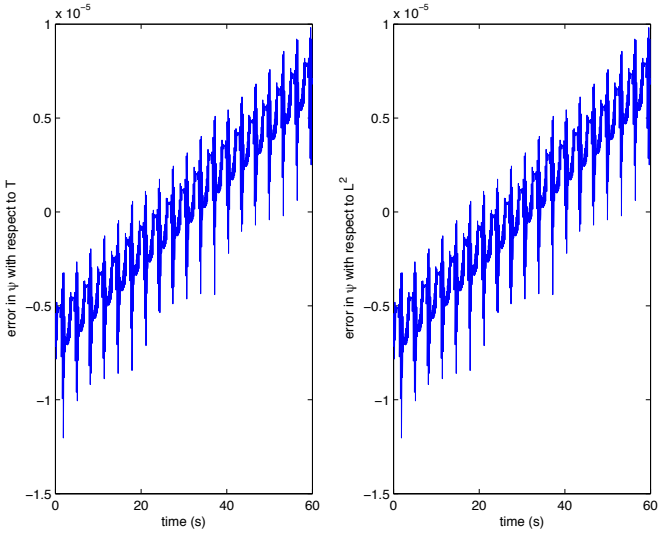

Figure A.26: Freely rotating rigid body; grid sampling method; 6th order polynomial; 1st motion integral.

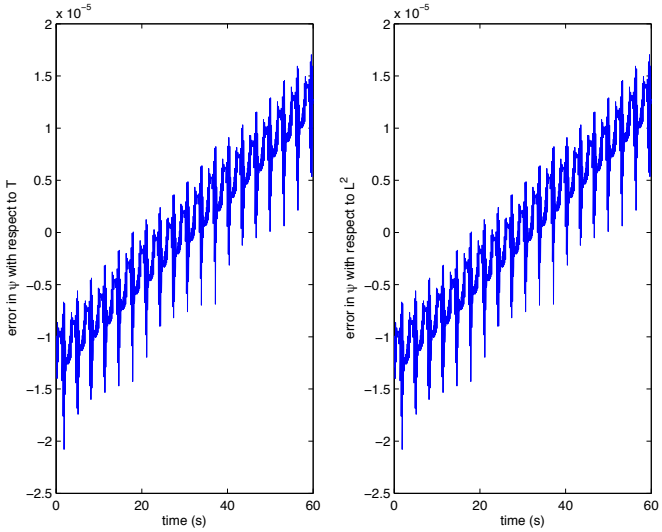

Figure A.27: Freely rotating rigid body; grid sampling method; 6th order polynomial; 2nd motion integral.

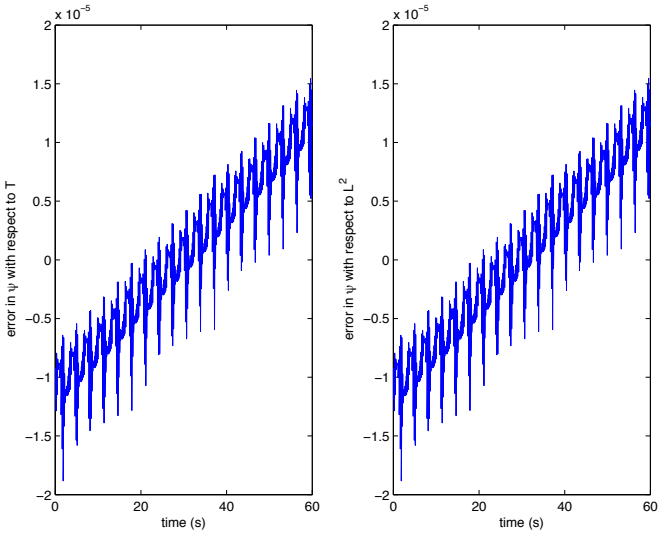

Figure A.28: Freely rotating rigid body; grid sampling method; 12th order polynomial; 1st motion integral.